Systematic Stress Testing and Model Risk

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Modelling in Life Insurance Conference
ISFA, Université Lyon 1
7 October 2015
1 First Generation Stress Tests

2 Second Generation Stress Tests

3 Measuring Model Risk: Stress Tests with Mixed Scenarios
Outline

1. First Generation Stress Tests
2. Second Generation Stress Tests
3. Measuring Model Risk: Stress Tests with Mixed Scenarios
Purpose of Stress Testing:
Complement statistical risk measurement

- Stress Tests: Which scenarios lead to big losses?
  Derive risk reducing action.
  (Statistical risk measurements: What are prob’s of big losses?)
- Stress Tests: Address model risk.
  Consider alternative risk factor distribution.
  (Statistical risk measurement: Assume fixed model.)
Requirements on stress scenarios (Basel II)

- plausible
- severe
- suggestive of risk reducing action

See Basel Principles of Sound Stress Testing
Reference risk factor distribution $\nu$, Portfolio loss function $L$, both on risk factor space $\Omega \subset \mathbb{R}^n$. 
What is a model?

**REALITY**
A loss $X$ occurs in a particular situation

**MODEL**

$X \sim L(r)$

$L$: payoff function of portfolio:
- fully controllable

$r$: Risk Factors:
- not controllable, observable
- not controllable, not observable

use of model changes reality
Distribution model risk

REALITY
A loss \( X \) occurs in a particular situation

MODEL
\[ X = L(r) \]

L: payoff function of portfolio:
- fully controllable

r: Risk Factors:
- not controllable, observable

use of model changes reality

specify distribution class
estimate parameters

estimated risk factor distribution
First Generation Stress Tests: Hand-picked Point Scenarios

- **Point scenario:** each risk factor gets a value: \( r \in \Omega \)
- A small number of scenarios is picked by hand, ideally involving heterogeneous groups of experts.
  
  \[ A = \{ r^1, r^2, \ldots \} \subset \Omega \]

  a small set of hand-picked scenarios.

- Find worst case scenario and worst case loss in \( A \)
  
  \[ \max_{r \in A} L(r) \]

- Worst case loss over \( A \) is a coherent risk measure.
First Generation Stress Tests: Examples

- most stress tests of market or credit risk performed by financial institutions
- SPAN rules
- FSAP stress tests
- US institutional stress tests during 2009 crisis
- 2014 stress tests of ECB
Criticism of First Generation Stress Tests

Are there any real stress tests whose results forced a bank to change strategy?

Accidental or deliberate misrepresentation of risks:

1. Neglecting severe but plausible scenarios
2. Considering too implausible scenarios
1 First Generation Stress Tests

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Second Generation Stress Tests: Plausible Scenarios

- **Measure of plausibility** for point scenarios:

  \[ \text{Maha}(r) := \sqrt{(r - \mathbb{E}(r))^T \cdot \Sigma^{-1} \cdot (r - \mathbb{E}(r))}, \]

  where \( \Sigma \) is covariance matrix of risk factor distribution \( \nu \).

- **Intuition:**
  Scenarios in which some risk factors move many standard deviations are implausible.
  Scenarios in which some pair of risk factors moves against their correlation are implausible.
Second Generation Stress Tests: Systematic Point Scenario Analysis

- Set of plausible scenarios

\[ A := \text{Ell}_h := \{ r : \text{Maha}(r) \leq h \}, \]

where \( h \) is the plausibility threshold.

- Systematic search of worst case scenario:

\[ \max_{r \in \text{Ell}_h} L(r) \]
Second generation stress of linear portfolio

• Loss linear function of $n$ normal risk factors:
  \[ L(r) = l^T(\mu - r), \nu \sim N(\mu, \Sigma). \]

• Systematic search of worst case scenario:
  \[ \max_{r \in \text{Ell}_h} l^T(\mu - r). \]

• Worst case scenario: \[ \bar{\mu} = \mu - \frac{h}{\sqrt{l^T \Sigma l}} \Sigma l \]

• Worst case loss: \[ \mathbb{E}_Q(L) = h \sqrt{l^T \Sigma l}. \]
Advantages of Systematic Stress Testing with Point Scenarios

All three requirements on stress testing are met:

- Do not miss plausible but severe scenarios.
- Do not consider scenarios which are too implausible.
- Worst case scenario over $\mathbb{E}_{\mathbb{l}_h}$ gives information about portfolio structure and suggests risk reducing action.
Problems of Systematic Stress Testing with Point Scenarios

1. What if risk factor distributions $\nu$ is non-elliptical?
2. What if risk true factor distribution is not $\nu$? Model risk is not addressed.
3. Maha does not take into account fatness of tails.
4. $\text{MaxLoss}_{\text{Ell}_k}$ depends on choice of coordinates.
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**Mixed Scenarios**

**Mixed scenario:** Probability distribution of point scenarios.

- **Interpretation 1:**
  Risk factor distributions alternative to the prior $\nu$. Model risk.

- **Interpretation 2:**
  Generalisation of point scenarios, but support not concentrated on one point.
Plausibility of Mixed Scenarios

- **Measure of plausibility for mixed scenarios:**
  relative entropy from \( \nu \)
  \((I\text{-divergence, information gain, Kullback-Leibler distance})\)

\[
I(Q||\nu) := \begin{cases} 
\int \frac{dQ}{d\nu}(r) \log \frac{dQ}{d\nu}(r) d\nu(r) & \text{if } Q \ll \nu \\
+\infty & \text{if } Q \not\ll \nu
\end{cases}
\]

- **Intuition:**
  Relative entropy \( I(Q||\nu) \) measures the ‘distance’ of the distributions \( Q \) and \( \nu \).
  \( I(Q||\nu) = 0 \) if and only if \( Q = \nu \) (as distributions)
Worst Case Scenario

- **Set of plausible scenarios:** Instead of ellipsoid take Kullback-Leibler sphere in the space of distributions

\[ A := S(\nu, k) := \{ Q : I(Q|\nu) \leq k \}. \]

- **Severity of scenarios:** Instead of \( L(r) \) take \( \mathbb{E}_Q(L) \)

- Systematic stress test with mixed scenarios:

\[ \sup_{Q \in S(\nu,k)} \mathbb{E}_Q(L) =: \text{MaxLoss}_k(L) \]

If it exists, call scenario achieving MaxLoss: \( \bar{Q} \).
Model Risk

- estimation error: wrong distribution parameters
- model misspecification: wrong model class
- \( \sup_{Q \in S(\nu,k)} \mathbb{E}_Q(L) \) quantifies effects of both on expected loss.
Advantages of Systematic Stress Testing with Mixed Scenarios

1. Scenario set is naturally defined for non-elliptical risk factor distributions $\nu$.

2. Model risk is addressed:
   Mixed scenarios are alternatives to prior risk factor distribution $\nu$.

3. Relative entropy does take into account fatness of tails of $\nu$.

4. $\text{MaxLoss}_k$ does not depend on choice of coordinates.
The Basic Tool

• Tool from large deviations theory for solving explicitly the optimisation problem \( \sup_{Q \in S(\nu, k)} E_Q(L) \):

\[ \Lambda(\theta) := \log \left( \int e^{\theta L(r)} d\nu(r) \right). \]
Basic Properties of the $\Lambda$-function

- $\Lambda(0) = 0$.
- $\Lambda$ is convex.
Solution of Worst Case: The Generic Case

Theorem

- Except in the pathological cases (i), (ii), (iii) below, the equation
  \[ \theta \Lambda'(\theta) - \Lambda(\theta) = k, \]  
  has always a unique positive solution \( \bar{\theta} \).
- The mixed worst case scenario \( \overline{Q} \) is the distribution with \( \nu \)-density
  \[ \frac{dQ}{d\nu}(r) = e^{\bar{\theta}L(r) - \Lambda(\bar{\theta})}, \]  
- The Maximum Loss achieved in the mixed worst case scenario \( \overline{Q} \) is
  \[ \mathbb{E}_{\overline{Q}}(L) = \Lambda'(\bar{\theta}). \]
Practical Calculation of Worst Case

1. Calculate $\Lambda(\theta)$. (Evaluate $n$-dimensional integral.)
2. Starting from the point $(0, -k)$, lay a tangent to $\Lambda(\theta)$ curve.
3. Worst case loss is given by the slope of the tangent.
4. Worst case scenario is distribution with density
   \[ \frac{dQ}{d\nu}(r) = e^{\bar{\theta}L(r) - \Lambda(\bar{\theta})}, \]
   where $\bar{\theta}$ is $\theta$-coordinate of tangent point.
Normal risk factors, linear portfolio

- Loss linear function of $n$ risk factors: $L(r) = l^T(\mu - r), \nu \sim N(\mu, \Sigma)$.
- $\Lambda$ quadratic: $\Lambda(\theta) = l^T \Sigma l \theta^2 / 2$.
- Worst case scenario: $\overline{Q} \sim N(\overline{\mu}, \Sigma)$ with $\overline{\mu} = \mu - \frac{h}{\sqrt{l^T \Sigma l}} \Sigma l$ where $h = \sqrt{2k}$.
- Worst case loss: $\mathbb{E}_{\overline{Q}}(L) = h \sqrt{l^T \Sigma l}$.
Summary

Systematic stress tests with mixed scenarios

- do not neglect dangerous scenarios when they are plausible,
- do not produce highly implausible scenarios,
- are applicable to both continuous and discrete risk factors with arbitrary distributions,
- quantify the effects of model risk...

... and can be implemented straightforwardly.
Some references


• Breuer T., M. Jandacka, K. Rheinberger, M. Summer: How to find plausible, severe, and useful stress scenarios, International Journal of Central Banking 5 (2009), 205-224
