Exit Probabilities and Balayage of Constrained Random Walks

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Abstract

Let X be the constrained random walk on $\mathbb{Z}_+^d$, $d \in \{2, 3, 4, ...\}$, representing the queue lengths of a stable Jackson network and let $x \in \mathbb{Z}_+^d$ be its initial position (X is a random walk with independent and identically distributed increments except that its dynamics are constrained on the boundaries of $\mathbb{Z}_+^d$ so that X remains in $\mathbb{Z}_+^d$; stability means that X has a nonzero drift pushing it to the origin). Let τ_n be the first time when the sum of the components of X equals n. The probability $p_n \doteq P_x(\tau_n < \tau_0)$ is one of the key performance measures for the queueing system represented by X and its analysis/computation received considerable attention over the last several decades. The stability of X implies that p_n decays exponentially in n. Currently the only analytic method available to approximate p_n is large deviations analysis, which gives the exponential decay rate of p_n. Finer approximations are available via rare event simulation. The present article develops a new method to approximate p_n and related expectations. The method has two steps: 1) with an affine transformation, move the origin to a point on the exit boundary associated with τ_n; let $n \to \infty$ to remove some of the constraints on the dynamics of the walk; the first step gives a limit *unstable /transient* constrained random walk Y 2) construct a basis of harmonic functions of Y and use this basis to apply the classical superposition principle of linear analysis (the basis functions can be seen as perturbations of the classical Fourier basis). The basis functions are linear combinations of log-linear functions and come from solutions of *harmonic systems*; these are graphs with labeled edges whose vertices represent points on the interior *characteristic surface* $\mathcal{H}$ of Y; the edges between the vertices represent conjugacy relations between the points on the characteristic surface, the loops (edges from a vertex to itself) represent membership in the boundary characteristic surfaces. Characteristic surfaces are algebraic varieties determined by the distribution of the unconstrained increments of X and the boundaries of $\mathbb{Z}_+^d$. Each point on $\mathcal{H}$ defines a harmonic function of the unconsrained version of Y. Using our method we derive explicit, simple and almost exact formulas for $P_x(\tau_n < \tau_0)$ for X representing d-tandem queues, similar to the product form formulas for the stationary distribution of X. The same method allows us to approximate the Balayage operator mapping f to $x \to \mathbb{E}_x\left[f(X_{\tau_n})1_{\{\tau_n<\tau_0\}}\right]$ for a range of stable constrained random walks representing the queue lengths of a queueing system with two nodes (i.e., $d = 2$). We provide two convergence theorems; one using the coordinates of the limit process and one using the scaled coordinates of the original process. The latter is given for two tandem queues (i.e., when the set of possible increments of X is $\{(0, 1), (-1, 1)(0, -1)\}$) and uses a sequence of subsolutions of a related Hamilton Jacobi Bellman equation *on a manifold*; the manifold consists of three copies of $\mathbb{R}_+^2$, the zeroth glued to the first along $\{x : x(1) = 0\}$ and the first to the second along $\{x : x(2) = 0\}$. We indicate how the ideas of the paper relate to more general processes and exit boundaries.

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