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ÉCOLE DOCTORALE SCIENCES ÉCONOMIQUES ET DE GESTION

TAIL DISTRIBUTION OF THE SUMS OF REGULARLY VARYING RANDOM VARIABLES, COMPUTATIONS AND SIMULATIONS

THÈSE

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Quang Huy NGUYEN

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Directeurs de thèse:	**Christian ROBERT**, Professeur à l'Université Lyon 1
Rapporteurs:	**Enkelejd HASHORVA**, Professeur à l'Université de Lausanne **Etienne MARCEAU**, Professeur à l'Université Laval
Examinateurs:	**Frédéric PLANCHET**, Professeur à l'Université Lyon 1 **Didier RULLIERE**, Maître de conférences à l'Université Lyon 1 **Olivier LOPEZ**, Maître de conférences à l'ENSAE

Résumé

Queues de distribution des sommes de variables aléatoires à variations régulières: calculs et simulation.

Cette thèse s'intéresse à l'utilisation de techniques numériques par approximation sous forme de séries et de techniques de simulation pour l'approximation de la queue de distribution de sommes de variables aléatoires à variations régulières. Le calcul de la probabilité que la somme soit plus grande qu'un seuil donné est important en gestion des risques. En particulier, ce calcul est utilisé pour définir le besoin en capital des sociétés d'assurances ou d'autres institutions financières.

Le premier chapitre constitue l'introduction de la thèse. Il explique les principaux résultats et présente les outils mathématiques qui sont développés dans la thèse.

Le second chapitre est basé sur le travail : "Series expansions for the sum of the independent Pareto random variables", article rédigé avec le Professeur Christian ROBERT, directeur de la thèse. Cet article est soumis à publication. Il propose un algorithme de calcul pour déterminer la queue de distribution d'une somme de variables aléatoires de type Pareto non nécessairement équidistribuées. Il propose une approximation sous forme de série de la fonction de survie de la somme. L'algorithme utilisé pour calculer l'approximation est simple, facile à implémenter, et offre de très bons résultats numériques.

Le troisième chapitre de cette thèse est basée sur l'article : "New efficient estimators in rare event simulation with heavy tails", publié dans Journal of Computational and Applied Mathematics, et co-écrit avec le Professeur Christian ROBERT. Il s'intéresse à l'approximation par simulation de la probabilité que la somme de variables aléatoires indépendantes à variations régulières soit plus grande qu'un seuil élevé. Des estimateurs efficaces ont déjà été introduits dans la littérature associée à la simulation d'événements rares. Nous proposons de nouvelles techniques de simulation qui sont plus efficaces que les méthodes précédemment proposées.

Le quatrième chapitre poursuit l'analyse de la simulation d'événements rares du type "la somme est plus grande qu'un seuil", mais cette fois-ci il s'intéresse à des situations où les variables aléatoires sont dépendantes. Il se focalise sur le cas où la dépendance est donnée par une copule archimédienne. Ce chapitre est basé sur l'article en relecture : "Efficient simulation of tail probabilities of sums with heavy tailed random variables and Archimedean copulas". Les équivalents asymptotiques de la probabilité de dépassement de seuil ne sont connus que dans des cas particuliers et ils fournissent en général des approximations très médiocres de la vraie valeur. Les techniques de simulation sont donc très appréciables pour obtenir rapidement des approximations précises. Nous proposons quatre estimateurs et quatre techniques de simulation

associées. Nous montrons que les erreurs relatives sont asymptotiquement bornées pour presque tous les estimateurs. Les simulations montrent que certains estimateurs sont plus précis.

Abstract

Tail distribution of the sums of regularly varying random variables, computations and simulations

This thesis aims to study computation and simulation methods to approximate tail distribution of the sums of regularly varying random variables. The paper proceeds as follows:

The first chapter provides the general introduction of the thesis.

The second chapter is essentially constituted by the article *"Series expansions for the sum of the independent Pareto random variables"* which was co-written with Professor Christian ROBERT, actually submitted for publication. It deals with the problem of estimating tail distribution of the sum of independent Pareto variables. This problem has been studied for a long time but a complete solution has not yet been found. In this section, we acquire an exact formula, a series expansions, for the distribution of the sum of independent Pareto of non-integer tail indices. Not only is this formula simple and easy to apply but it also gives better numerical results than most of existing methods.

The third chapter rests on the article *"New efficient estimators in rare event simulation with heavy tails"*, co-written with Professor Christian ROBERT, currently published on *"Journal of Computational and Applied Mathematics 261, 39-47"* in 2013. Practically, efficient estimation for tail distribution of the sum of i.i.d. regularly varying random variables is one of widely researched problems in rare event simulation. In this context, Asmussen and Kroese's estimator has performed better than other works. This part will introduce a new way to approach the sum. Our obtained estimator is more efficient than Asmussen and Kroese's estimator in the case of regularly varying tail. In other cases, combined with techniques of conditional Monte Carlo and importance sampling, our estimator is still better.

In the fourth chapter, we continue to study the tail behavior of the sum of regularly varying variables, with additional assumption that the dependence follows an Archimedean copula or an Archimedean survival copula. This section hinges on the article *"Efficient simulation of tail probabilities of sums with heavy tailed random variables and Archimedean copulas"* which is under consideration for being published. Almost all previous studies on this problem used asymptotic approaches which are hard to control the errors. Therefore, techniques of simulation to calculate the tail probability of the sum are presented. Though some of our estimators have bounded relative errors while the others do not, all of them give favorable numerical performances for such a challenging problem.

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For Thao Nguyen

Chapter 1

General introduction

This thesis focuses on calculating the probability that the sum of regularly varying positive risks exceeds a large threshold. We obtain results from the simple case in which the risks are independent Pareto random variables to the complex case where the risks are regularly varying random variables, and the dependence follows an Archimedean model.

1.1 Presentation of the thesis

This dissertation contains four chapters organized as follows:

The first chapter entitled "General introduction", gives an overview of the existing works related to the thesis and briefly presents our results.

The second chapter entitled "Exact computation for the sums of independent Pareto variables" is co-authored with Prof Christian ROBERT and is currently being submitted for publication under the title *"Series expansions for the sums of independent Pareto random variables"*. This section aims to introduce a construction of an exact formula for the distribution of the sums of independent Pareto random variables. Although Pareto distribution is simple, the distribution of the sum of two or more independent Pareto random variables is not less simple than the distribution of the sum of other regularly varying random variables. In particular, very few explicit analytical expressions of the convolutions are known, for example: Hagstroem (1960)[28] ; Brennan, Reed and Sollfrey (1968)[20]; Blum (1970)[18]; Ramsay (2006) [45]; and Ramsay (2008) [46] but all these works are limited due to strong assumptions for the parameters of Pareto variables. Some results of the sums of independent regularly varying variables

are applicable to this part's problem: Geluk, Peng and De Vries (2000)[25]; Barbe and McCormick (2005)[15] and Albrecher, Hipp and Kortschak (2010)[1]. However, these papers study asymptotic behaviors of the sum rather than exact approaches.

In this chapter, we decide to focus on the sums of independent Pareto random variables. The major outcome provides an infinite series expansions in the case of Pareto random variables with non-integer tail indices. The advantage of our finding is that the Pareto variables may have different tail indices and different scale parameters. This formula is applicable in calculating risk measures such as VaR, TVaR, ES, etc... for portfolios of independent Pareto risks and deriving an exact formula for compound sums of Pareto claims.

The third chapter entitled "The sums of i.i.d. regularly varying random variables and simulation" is co-written with Prof Christian ROBERT and is recently published under the title *"New efficient estimators in rare event simulation with heavy tails"*, on *"Journal of Computational and Applied Mathematics 261, 39-47"*. This section is about the efficient simulation of probability $z(s) = P(S_n > s)$ where s takes a large value and S_n is the sum of n i.i.d. regularly varying random variables $X_1, \cdots, X_n$.

The classical Monte Carlo simulation method (so-called crude Monte Carlo) is not efficient in estimating small or very small probabilities because the variance of the estimators is much larger than the expectation. Methods of variance reduction for estimators of small probabilies as $z(s)$ have been studied for a long time. For example, Asmussen and Binswanger (1997)[4] proposed conditional Monte Carlo methods; Asmussen, Binswanger and Hojgaard (2000)[11] and Juneja and Shahabuddin (2002)[40] used importance sampling techniques; Dupuis, Leder, and Wang (2007)[23] and Hult and Svensson (2012)[36] provided the dynamic importance sampling estimators. The best of existing estimators for $z(s)$ was proposed by Asmussen and Kroese (2006)[9] (called $Z_{AK}(s)$ in this thesis). This estimator is shown in Asmussen and Kroese's to have bounded relative error and in Hartinger and Kortschak (2009)[35] to have vanishing relative error. The exact rate of decay of the relative error of this estimator has been recently given by Asmussen and Kortschak (2012)[9] under the assumption that probability density functions of marginal variables exist.

In this chapter, we introduce a new way to construct efficient estimators for $z(s)$. We use a binomial random variable to control the sums of i.i.d. regularly varying variables. If the tail index of marginal variables is less than 1, our first estimator performs better than Asmussen and Kroese's one without any development. In other cases, combined with techniques of conditional Monte Carlo (when the tail index is less than 2) or importance sampling (when the tail index is larger than 2), our estimator still has better results.

The fourth chapter: "The sums of dependent regularly varying variables and simulation" is co-authored with Prof Christian ROBERT and presently under the provision of being published under the title *"Efficient simulation of tail probabilities of sums with heavy tailed random variables and Archimedean copulas"*. In this part, we calculate tail distribution of the sums of regularly varying random variables via simulation techniques. The dependence between the variables is assumed to follow an Archimedean copula or an Archimedean survival copula.

All past research related to this problem are developed under asymptotic approaches. Albrecher et al. (2005)[3], Jessen and Mikosch (2006)[37] and Yuen and Yin (2012)[51] studied the relation between the tail of sums and the tail of marginal variables. Wuthrich (2003)[50] and Sun and Li (2010)[49] considered Archimedean copulas for the dependence and worked out results in integral form. Techniques of simulation are used to calculate tail probabilities of dependent sums in Blanchet et al. (2009)[17], Kortschak and Hashorva (2013)[41], but only for the case of log-elliptic vectors. There are not many studies relating to the sums of regularly varying variables under Archimedean copula. In general, the problems of calculating tail distribution of a dependent sum include:

- *How to simulate the risks under dependent assumption*: this is a challenging question since it is hard to calculate the conditional distributions and their inverses using classical method. Therefore, a dependence structure is often simulated by means of special stochastic representations, the Gaussian vector is an example of such approach.
- *What technique to reduce the variance*: even if the dependence structure can be simulated, the variance of classical Monte Carlo method is too large. To get an efficient estimator, it requires some variance reduction techniques.

In the fourth chapter, we use the stochastic representation of McNeil (2009)[42] and the simulating procedure of Brechmann (2013)[19] to simulate the dependence structure of Archimedean copula of generator Φ and focus on the conditional Monte Carlo method to construct some estimators with efficient relative error for the probability that the sum of regularly varying variables under Archimedean or Archimedean survival copula is over a large value. Our approach has the advantage that it does not require the assumption of identically distributed of marginal variables.

1.2 The main results

In this section, a brief summary of each chapter is presented with the introduction of existing works followed by our contribution conclusions.

1.2.1 Exact computation for the sums of independent Pareto variables

Convolution of independent Pareto risks

The Pareto distributions are a primary subclass that is traditionally used by actuaries to model catastrophic losses in an insurance portfolio, by banks' risk managers to quantify operational risk (the risk of losses resulting from inadequate or failed internal processes, people and systems, or external events) and even by economists to model the income distribution of populations etc,... These distributions also appear widely in physics, biology, earth and planetary sciences, computer science, demography and social sciences.

Definition 1.1. *If X is a Pareto random variable then the probability that X is greater than x, i.e the survival distribution function is given, for $x \geq \beta$, by*

$$\mathbb{P}(X > x) = \mathbb{I}_{\{x \geq \beta\}} \left(\frac{\beta}{x}\right)^{\alpha}$$

where $\alpha > 0$ is called the tail-index and $\beta > 0$ is called the scale parameter.

There are different ways of defining Pareto variables. Another way, for example, is given by the Lomax random variables.

Definition 1.2. *If X is a Lomax random variable with scale parameter β and tail-index α then the probability that X is greater than x, i.e the survival distribution function is given, for $x \geq 0$, by*

$$\mathbb{P}(X > x) = \left(\frac{\beta}{\beta + x}\right)^{\alpha}.$$

Clearly, if X and Y are Pareto random variable and Lomax random variable of the same parameters (α, β) respectively, then we have $X \overset{d}{=} Y + \beta$. Suppose that we have n independent risks in the portfolio denoted by $X_1, X_2, \cdots, X_n$ and all follow a Pareto distribution of parameters $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_n, \beta_n)$ respectively. We are interested in the problem of calculating the probability that sum of the variables exceeds a large threshold $\mathbb{P}(X_1 + X_2 + \cdots + X_n > s)$. As we mentioned in Section 1.1, Pareto distribution has a simple distribution function, however, the distribution of the sum of n independent variables is difficult to obtain. One of the first solution for this problem is developed in Hagstroem (1960)[28]. The author derived the exact results for the case where $\alpha_i = \beta_i = 1$ for $i = 1, 2, \cdots, n$ and $n \leq 3$.

Proposition 1.3 (Hagstroem, 1960)**.** *If* $p_n(s) = \mathbb{P}(\sum_{i=1}^{n} X_i > s)$ *with* $\alpha_i = \beta_i = 1$ *for* $i = 1, 2, \cdots, n$ *and* $n \leq 3$*, then*

$$\begin{aligned} p_1(s) &= \frac{1}{s} \text{ for } s \geq 1, \\ p_2(s) &= \frac{2}{s} + \frac{\log(s-1)}{s^2} \text{ for } s \geq 2, \end{aligned}$$

and for $s > 3$,

$$\begin{aligned} p_3(s) &= \frac{3}{s} + \frac{6(s-2)\log(s-2)}{s^3(s-1)} + \frac{4}{s^3}\log(s-1)\log(s-2) \\ &+ \frac{2}{s^3}\left[\big(\log(x-1)\big)^2 - \big(\log 2\big)^2\right] - \frac{4}{s^3}\left[L\left(\frac{1}{s-1}, \frac{1}{s-1}\right) + L\left(\frac{1}{s-1}, \frac{1}{2}\right)\right], \end{aligned}$$

for $0 < a, b < 1$*, and* $a + b < 1$*, where*

$$L(a,b) = \int_{b}^{1-a} \frac{-\log x}{1-x} dx.$$

We can make a remark that even with strong assumptions: $\alpha_i = \beta_i = 1$ and $n \leq 3$, the survival distribution function of the sum of three variables still becomes much more complex. There is no result for $p_n(s)$ when $n \geq 4$. In 1970, Blum derived a series expansion for the density function of S_n for the case $\beta_i = 1$ and $\alpha_i = \alpha \in (0,2), \alpha \neq 1$ for $i = 1, 2, \cdots, n$.

Proposition 1.4 (Blum, 1970)**.** *If* f_n *is the pdf of* S_n *with* $\beta_i = 1$ *and* $\alpha_i = \alpha \in (0,2), \alpha \neq 1$ *for* $i = 1, 2, \cdots, n$ *then*

$$f_n = -\frac{1}{\pi}\sum_{i=1}^{n} C_n^i [-\Gamma(1-\alpha)]^i \ \sin(\pi\alpha i) \sum_{m=0}^{\infty} \xi_{n-i,m} \frac{\Gamma(m+\alpha i+1)}{x^{m+\alpha i+1}},$$

where $\xi_{k,m}$ *is the* m^{th} *coefficient in the series expansion of the* k^{th} *power of the confluent hypergeometric function* ${}_1F_1(-\alpha, 1-\alpha, t) = \sum_{j=0}^{\infty} \frac{1}{j!} t^j$ *i.e* $\sum_{j=0}^{\infty} \xi_{k,m} t^m = \big({}_1F_1(-\alpha, 1-\alpha, t)\big)^k$.

The exact expressions for the density of the sum of n independent Pareto random variables by Hagstroem (1960) and Blum (1970) are only valid for the identical marginal distributions and a small range of parameters, namely, $\alpha = 1$ and $n \leq 3$ or $0 < \alpha < 2$; $\alpha \neq 1$. Blum cautioned that it may be difficult to compute with some values of s, of α, and when n is large. Recently, Ramsay (2006) has developed the exact formula for distribution of the sum of independent Pareto random variables. His formula is an extension of previous study of Hagstroem (1960)

and it covers the case where the Pareto random variables are i.i.d. of integer tail-index. His final result is in an integral form.

Proposition 1.5 (Ramsay, 2006)**.** *If F_n and f_n are the distribution and density of S_n with $\beta_i = \beta$ and $\alpha_i = m \in \mathbb{N}$ then*

$$f_n(s) = \frac{1}{n\beta} \int_0^\infty \exp\left[-(1 + \frac{s}{n\beta})u\right] \ \varphi_{m,n}(u/n)\ du,$$

$$F_n(s) = \int_0^\infty \frac{1}{u} \ \left[1 - \exp\big(-\frac{su}{\beta}\big)\right] \ \exp(-nu)\ \varphi_{m,n}(u/n)\ du,$$

where

$$\varphi_{m,n}(u) = (-1)^{n+1} m^n \sum_{j=0}^{[(n-1)/2]} (-\pi^2)^j \ C_n^{2j+1} \ (Ei_{m+1}(u))^{n-2j-1} \left(\frac{u^m}{m!}\right)^{2j+1},$$

$$Ei_{m+1}(u) = \frac{u^m}{m!}\left[\gamma + \log u - \sum_{j=1}^{m} \frac{1}{j}\right] + \sum_{r=0, r\neq m}^{\infty} \frac{u^j}{(j-m)j!},$$

and $[t]$ denotes the greatest integer less than or equal to t and γ is the Euler constant.

Ramsay (2008)[46] expanded his work to the case where tail index is not integer and scale parameters are different. The final result also has an integral form.

Asymptotic approaches for the sums of i.i.d. regularly varying random variable

Pareto random variable is a special case of regularly varying random variables then all results on regularly varying variables can be applied to a Pareto random variable.

Definition 1.6. *A measurable function f, $f : [0, \infty) \to [0, \infty)$ is said to be regularly varying at ∞ of index $\vartheta, \vartheta \in \mathbb{R}$, if it satisfies*

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^\vartheta \ \textit{for all}\ t \in \mathbb{R}^+.$$

We write $f \in \mathcal{RV}_\infty(\vartheta)$. If $\vartheta = 0$ then f is said to be slowly varying.

Definition 1.7. *A random variable X is said regularly varying if there exists an index $\alpha > 0$ such that its survival distribution is a regularly varying function at ∞ with index $-\alpha$, i.e $\mathbb{P}(X > x) = \bar{F}_X(x) \in \mathcal{RV}_\infty(-\alpha)$.*

According to the Karamata's representation (see A.1), all regularly varying function $\bar{F} \in \mathcal{RV}_\infty(-\alpha)$ can be written as $\bar{F}(x) = x^{-\alpha}L(x)$ where $L(x)$ is a slowly varying function. If

the slowly varying function equals to a constant, we have the Pareto variable whose scale parameter equals to that constant. Albrecher, Hipp and Kortschak (2010)[1] derived a series expansion for distribution of the sum of n i.i.d. regularly varying variables of common survival function $\bar{F}(x)$, the error of the result is asymptotically bounded by $[\bar{F}(s)]^2$.

Theorem 1.8 (Albrecher, Hipp and Kortschak, 2010)**.** *Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables of common survival distribution function $\bar{F} \in \mathcal{RV}_{-\alpha}$. If the function $\bar{F}$ is $[\alpha]$-times continuously differentiable*
If $k < \alpha < k+1$ then

$$\mathbb{P}(S_n > s) = a_{k+1}(s) - C_n^2 \bar{F}(s)^2 \ (1-2\alpha) \ Beta(1-\alpha, 1-\alpha) + O(\bar{F}(s)^2).$$

If $\alpha = k+1$ then

$$\mathbb{P}(S_n > s) = a_{k+1}(s) + \frac{n(-1)^{k+1}\bar{F}^{(k+1)}(s)}{(k+1)!} \int_0^{s/2} x^{k+1} dF^{*(n-1)}(x) + O\left(\bar{F}^{(k+1)}(s) \int_0^{s/2} x^{k+1} dF^{*(n-1)}(x) \right)$$

where $a_1(s) = n\bar{F}(s)$ and $a_h(s) = a_1(s) + \sum_{j=1}^{h-1} \frac{n\mathbb{E}\left[(X_1 + \cdots + X_{n-1})^j\right]}{j!} \bar{F}^{(j)}(s)$; $h = 2, \cdots, k+1$.

Apply this formula to the sum of n i.i.d. Pareto random variables of parameters (α, β), we have

$$\mathbb{P}(S_n > s) = n\beta^\alpha \big(s^{-\alpha} + \sum_{j=1}^{k-1} \frac{\alpha \cdots (\alpha+j-1)}{j!} \sum_{\sum j_r = j}^{1 \le r \le j} \frac{j!}{j_1! \cdots j_h!} \prod_{l=1}^{h} \frac{\alpha \beta^{j_l}}{\alpha - j_l} s^{-\alpha-j} \big) + O\big(s^{-\alpha-k+1}\big)$$

where $k = [\alpha + 1]$. (see the Proof 1.3)

Inverse Laplace transform to approximate the sum of independent Paretos

Inverse Laplace transform is a numerical approach to approximate the distribution of the sum of independent Pareto random variables. This method is widely used in sciences with many applications of physics and engineering. If the Laplace transform of a function exists and is calculable, we can approximate that function via inverse techniques.

Definition 1.9. *The Laplace transform of a function $f(t)$, defined for all real numbers $t \ge 0$, written by $\mathcal{L}(f(t))$, defined by*

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-tx} f(x) dx.$$

Generally, inverse Laplace transform is applicable to calculate the distribution of sum of random variables when the random variables are independent because the Laplace transform of the sum

is calculated by the product of the Laplace transforms of these marginal variables. The Laplace transform of a Pareto with parameters (α_i, β_i) is determined by

$$\mathcal{L}(f_i(t)) \quad = \quad \mathbb{E}(e^{-tX_i}) = \alpha_i \beta_i^{\alpha} \int_{\beta_i}^{\infty} x^{-\alpha_i - 1}\ e^{-tx} dx = \alpha_i \beta_i^{\alpha_i}\ \Gamma(-\alpha_i, \beta_i t),$$

where $\Gamma(a, b)$ is upper incomplete gamma function. The Laplace transform of the sum is then

$$\mathcal{L}(f_{S_n}(t)) = \mathbb{E}(e^{-tS_n}) = \prod_{i=1}^{n} \mathbb{E}(e^{-tX_i}) = t^{\sum_{i=1}^{n} \alpha_i} \prod_{i=1}^{n} \alpha_i \beta_i^{\alpha_i} \Gamma(-\alpha_i, \beta_i t).$$

There are several methods proposed to calculate inverses Laplace transform. One of the most well-known is the Gaver-Wynn-Rho's (GWR) algorithm. If $f(t, M)$ is numerical result calculated by inverse Laplace transform and $f(t)$ is the true value, the error of GWR method satisfies

$$\frac{|f(t) - f(t, M)|}{|f(t)|} \leq 10^{-0.8M}$$

where parameter M of the algorithm is an even integer. For more details about inverse Laplace transform and GWR algorithm, see Appendix A.3.

Our result

In the second chapter, we present an analytic approach to calculate the distribution of the sum of independent Pareto variables. The Pareto variables can have different tail indices and scale parameters. The calculation technique used in this section is very simple. The result has a familiar form as a series expansion and it can be calculated directly if no tail indices are integer. In the case that there are some integer tail indices, with the high precision calculation, we can get approximate results. We try to develop an exact formula for the probability $\mathbb{P}(S_n > s)$ where the marginal variables are n Paretos: $X_1(\alpha_1, \beta_1), \cdots, X_n(\alpha_n, \beta_n)$. Our key result is organized as follow:

- If no tail indices $\alpha_1, \cdots, \alpha_n$ are integer, the probability $\mathbb{P}(S_n > s)$ is written in a series expansion form:

$$\mathbb{P}(S_n > s) = \sum^{\theta \in \Theta} c_\theta\ s^{-\alpha_\theta}$$

  where $\alpha_\theta \in \{\alpha_{i_1} + \cdots + \alpha_{i_j} + k : \{i_1, \cdots, i_j\} \in \{1, \cdots, n\}; k \in \mathbb{N}\}$.

- Corresponding coefficient of $s^{-\alpha_\theta}$ is calculated by the functions c, w and h, which will be determined easily. If there is a subset $\{i_1, \cdots, i_j\}$ of $\{1, \cdots, n\}$ such that $\alpha_{i_1} + \cdots + \alpha_{i_j} \in \mathbb{N}$

then all coefficients correspond to $s^{-\sum_{l=1}^{j}\alpha_{i_l}+k}$ are vanishing. That means there is no integer power in the series expansion.

- There is an exact formula for distribution of the sum of two Paretos of integer tail indices. If there are more than three integer indices, supposed to be $\alpha_{i_1}, \cdots, \alpha_{i_l}$, the calculation is done with the tail indices $\alpha_{i_1}+\epsilon, \cdots, \alpha_{i_l}+\epsilon$. The obtained result is a lower bound for $P(S_n > s)$. Similarly, the calculation with $\alpha_{i_1}-\epsilon, \cdots, \alpha_{i_l}-\epsilon$ gives us an upper bound for $P(S_n > s)$. When ϵ is close to 0, the upper bound and the lower bound converge to the true value (see Section 2.5).

Our formula in form of series expansion is the complete expression for distribution of the sum of independent Pareto variables of non-integer tail indices and different scale parameters. The limitation is obvious since if there is no exact formula in the case that there are three or more integer tail indices, the result is obtained numerically and the calculation requires high precision. In contrast, it has certain advantages over existing works: tail indices and scales parameters can be different, final formula is in familiar form, and distribution function can be calculated for any range of s and very quickly when s is large. Moreover, the exact formula allows us to have some interesting extensions. For example, we can derive a simple formula for the sum of N i.i.d. Pareto variables where N is a random variable. We can also calculate distribution of the sum of Pareto variables where the scale parameters are dependent random variables with finite expectations.

1.2.2 The sums of i.i.d. regularly varying random variables and simulation

This chapter targets on tail behavior of distribution of the sums of independent regularly varying random variables. The variables are assumed to be identically distributed with same survival distribution function $\bar{F} \in \mathcal{RV}_\infty(-\alpha)$. As mentioned above, regularly varying variable is a general case of Pareto where the scale parameter is replaced by a slowly varying function (see 1.6). Several techniques of Monte Calrlo simulation are applied to approach the sums. Indeed, the idea of these techniques comes from the weak law of large number: If $Y_1, Y_2, \cdots$ is an infinite sequence of i.i.d. random variables with expected value $\mathbb{E}(Y_1) = \mathbb{E}(Y_2) = \cdots = \mu$ then

$$\lim_{N\to\infty} \frac{Y_1 + Y_2 + \cdots + Y_N}{N} \overset{d}{=} \mu.$$

To calculate $z(s) = \mathbb{P}(S_n > s)$ via Monte Carlo simulation, we first create a random variable $Y(s)$ such that $\mathbb{E}(Y(s)) = z(s)$, then using a sample $(Y_1(s), Y_2(s), \cdots, Y_N(s))$ of N i.i.d. random variables of the same distribution function of $Y(s)$. Finally, $\bar{Y} = \dfrac{Y_1 + \cdots + Y_N}{N}$ is an approximate value of $z(s)$. In this case, variable $Y(s)$ is called an estimator for $z(s)$.

Crude Monte Carlo and infinite relative error

Crude Monte Carlo is the simplest way to construct an estimator for $\mathbb{P}(S_n > s)$. We denote this estimator as $Z_1(s)$. It is defined by

$$Z_1(s) = \mathbb{I}_{\{X_1+\cdots+X_n>s\}}.$$

By this definition, $Z_1(s)$ is a Bernoulli random variable of parameter $z(s)$. The expectation and variance of $Z_1(s)$ are: $\mathbb{E}(Z_1(s)) = z(s)$ and $\mathbb{V}ar(Z_1(s)) = z(s)(1-z(s))$. In the context of rare event simulation, the important criteria to evaluate quality of an estimator is relative error (so-called variation coefficient). It is defined by

Definition 1.10. *The relative error of an unbiased estimator $Z(s)$ of $z(s)$, denoted $e(Z(s))$, is the ratio of the standard deviation on the expected value*

$$e(Z(s)) = \frac{\mathbb{S}d(Z(s))}{\mathbb{E}(Z(s))} = \frac{\mathbb{S}d(Z(s))}{z(s)}.$$

We can verify the relative error of crude Monte Carlo estimator of $\mathbb{P}(S_n > s)$

$$e(Z_1(s)) = \frac{\sqrt{z(s)(1-z(s))}}{z(s)} = \sqrt{(1-z(s))/z(s)} \sim \sqrt{1/z(s)} \to \infty \quad when \quad s \to \infty.$$

Using crude Monte Carlo method, the ratio of standard deviation of $\bar{Y}$ on its expected value is asymptotically equivalent to $\sqrt{N/z(s)}$. The number of replications must be large enough compared to $[z(s)]^{-1}$ to have an efficient approximation. Practically, this method is ineffective when $z(s)$ is less than 10^{-4}. For probabilities smaller than 10^{-9}, it is impossible because of the problem of random number generator.

Conditional Monte Carlo and Asmussen and Kroese's estimator

In the following sections we will review variance reduction techniques for estimators. As for the effeciency of an estimator in rare event simulation, Asmussen and Glynn (2006) defined it in three levels of relative error.

Definition 1.11. *An unbiased estimator $Z(s)$ of the probability $z(s) = P(S_n > s)$ is called*

- *(1) Logarithmically efficient estimator if* $\limsup_{s\to\infty} e(Z(s))\ [z(s)]^{\epsilon} = 0$ *for all* $\epsilon > 0$.
- *(2) Estimator with bounded relative error if* $\limsup_{s\to\infty} e(Z(s)) < \infty$.

- *(3) Estimator with vanishing relative error* $\limsup_{s\to\infty} e(Z(s)) = 0$.

It is obvious that (3) is better than (2) and (1) is the lowest level of efficiency. Moreover, estimators with vanishing relative error can be evaluated from their rates of decay. For example, if $Z_a(s)$ and $Z_b(s)$ have vanishing relative errors satisfying $e(Z_a(s)) \sim [z(s)]^a$ and $e(Z_b(s)) \sim [z(s)]^b$ with $0 < a < b$, respectively, we say that $Z_b(s)$ is better than $Z_a(s)$.

A well-known variance reduction method in rare event simulation is conditional Monte Carlo. Asmussen and Glynn (2006)[7] defined a conditional Monte Carlo estimator for $\mathbb{P}(S_n > s)$ by: $\mathbb{P}(S_n > s|\mathcal{F})$ where $\mathcal{F}$ is a subset of the $\sigma - field\ \sigma(X_1, \cdots, X_n)$. The challenge, however, is how to find the condition $\mathcal{F}$ to minimize the variance (or relative error) of $\mathbb{P}(S_n > s|\mathcal{F})$. A very simple idea for $\mathcal{F}$ is $\mathcal{F} = \{X_1, \cdots, X_{n-1}\}$, which results in estimator called $Z_2(s)$:

$$Z_2(s) = \mathbb{P}(S_n > s|X_1, \cdots, X_{n-1}) = \bar{F}(s - S_{n-1})$$

where $S_{n-1} = X_1 + \cdots + X_{n-1}$. To obtain $Z_2(s)$, we simulate $(n-1)$ variables $X_1, \cdots, X_{n-1}$ instead of simulating all the variables. Variance of $Z_2(s)$ is smaller than that of crude Monte Carlo estimator $Z_1(s)$, but they are asymptotically equivalent. Indeed

$$\mathbb{E}\left([Z_2(s)]^2\right) = \mathbb{E}\left([\bar{F}(s - S_{n-1})]^2\right) \geq \mathbb{E}\left([\bar{F}(s - S_{n-1})]^2, X_1 > s\right) = \bar{F}(s) \sim \frac{z(s)}{n}.$$

Obviously, relative error of $Z_2(s)$ has no improvement compared to $Z_1(s)$

$$e(Z_2(s)) = \frac{\mathbb{S}d(Z_2(s))}{z(s)} \geq \frac{\sqrt{z(s)/n - [z(s)]^2}}{z(s)} = \sqrt{(1/n - z(s))/z(s)} \sim e(Z_1(s))/\sqrt{n}.$$

The relative error of $Z_2(s)$ is large because in the condition $\mathcal{F} = \{X_1, \cdots, X_{n-1}\}$, the probability that a single X_i takes a large value is too high. To get better performances, Asmussen and Glynn introduced the second conditional Monte Carlo estimator for the sums of i.i.d. regularly varying random variables. The condition $\mathcal{F}$ is developed from the order statistics of $\mathbf{X}$: suppose that vector $(X_1, X_2, \cdots, X_n)$ can be arranged in non-decreasing order: $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$, if $\mathcal{F} = \{X_{(1)}, X_{(2)}, \cdots, X_{(n-1)}\}$ we have the second conditional Monte Carlo estimator $Z_3(s)$:

$$Z_3(s) = \mathbb{P}(S_n > s|X_{(1)}, X_{(2)}, \cdots, X_{(n-1)}).$$

Note that if $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ is the order statistic of $(X_1, X_2, \cdots, X_n)$ then the survival distribution of $X_{(n)}$ conditioned on $X_{(1)}, \cdots, X_{(n-1)}$ is

$$\mathbb{P}(X_{(n)} > s|X_{(1)}, X_{(2)}, \cdots, X_{(n-1)}) = \frac{\bar{F}\left(s \vee X_{(n-1)}\right)}{\bar{F}(X_{(n-1)})}.$$

Therefore, we can write the estimator $Z_3(s)$ as follows

$$Z_3(s) = \mathbb{P}(X_{(n)} > s - S_{(n-1)}|X_{(1)}, X_{(2)}, \cdots, X_{(n-1)}) = \frac{\bar{F}\big(s - S_{(n-1)} \vee X_{(n-1)}\big)}{\bar{F}(X_{(n-1)})}.$$

Proposition 1.12 (Asmussen and Glynn, 2006)**.** *Conditional Monte Carlo estimator $Z_3(s)$ is logarithmically efficient i.e $e(Z_3(s))[z(s)]^\epsilon < 0$ for all $\epsilon > 0$.*

The relative error of $Z_3(s)$ meets the first requirement of efficiency. Asmussen and Kroese (2006)[10] used the property that variables are exchangeable and symmetry in a sum to introduce another conditional Monte Carlo estimator for $z(s)$. The idea here is partitioning according to which X_i is the largest i.e for which i one has $M_n = X_{(n)} = X_i$, and conditioning on the X_j with $j \neq i$. The estimator of Asmussen and Kroese, called $Z_4(s)$ in the book of Asmussen and Glynn[7], and $Z_{AK}(s)$ in this thesis, is defined by

$$Z_{AK}(s) = n\,\mathbb{P}\left(S_n > s, X_n = M_n|X_1, \cdots, X_{n-1}\right) = n\bar{F}\left(M_{n-1} \vee (s - S_{n-1})\right).$$

It can be seen that $Z_{AK}(s)$ has bounded relative error. Indeed, if $M_{n-1} \leq s/n$ then $S_{n-1} \leq (n-1)s/n$. Therefore we have $M_{n-1} \vee (s - S_{n-1}) \geq s/n$. Thus, it follows that

$$\mathbb{E}\left([Z_{AK}(s)]^2\right) \leq n^2\left([\bar{F}(s/n)]^2\right) \sim n^{2+2\alpha}[\bar{F}(s)]^2 \sim n^{2\alpha}[z(s)]^2.$$

Hartinger and Kortschak (2009)[35] proved that the performance of this estimator is better than the bounded relative error i.e it has vanishing relative error. Recently, the exact rates of decay of $e(Z_{AK}(s))$ have been given by Asmussen and Kortschak (2012)[9] with the assumption that the probability density function f of X_1 exists.

Theorem 1.13 (Assmusen and Kortschak, 2012)**.** *With assumption that the density of the marginal variables exists and are regularly varying: $f(x) = \alpha x^{-(\alpha+1)}l(x)$;*

- *If $\alpha > 2$ or $\mathbb{E}(X_1^2) < \infty$, then $\mathbb{V}ar(Z_{AK}(s)) \sim n^2(n-1)\mathbb{V}ar(X_1)[f(s)]^2$.*
- *If $\alpha = 2$ and $\mathbb{E}([X_1]^2) = \infty$, then $\mathbb{V}ar(Z_{AK}(s)) \sim 2n^2(n-1)[f(s)]^2 \int_0^s u\bar{F}(u)du$.*
- *If $\alpha < 2$, then $\mathbb{V}ar(Z_{AK}(s)) \sim n^2(n-1)\ k_\alpha\ [\bar{F}(s)]^3$ where*

$$k_\alpha = 2^\alpha + \frac{2^{3\alpha}}{3} - 2^{2\alpha} + \alpha\int_0^{1/2}((1-u)^{-\alpha} - 1)^2u^{-\alpha-1}du = \alpha\int_0^\infty\left[((1-u)\vee u)^{-\alpha} - 1\right]^2u^{-\alpha-1}du.$$

The exact rate of decay for $e(Z_{AK}(s))$ depends from α:

- For $\alpha > 2$ or more general, $\mathbb{E}(X_1) < \infty$

$$e(Z_{AK}(s)) = \frac{\mathbb{S}d(Z_{AK}(s))}{z(s)} \sim \frac{\sqrt{n-1}\ \mathbb{S}d(X_1)\ f(s)}{\bar{F}(s)} = (1+o(1))\ \alpha\sqrt{n-1}\ \mathbb{S}d(X_1)\ s^{-1}.$$

- For $\alpha = 2$ and $\mathbb{E}([X_1]^2) = \infty$

$$e(Z_{AK}(s)) \sim \frac{f(s)\left[2(n-1)\int_0^s u\bar{F}(u)du\right]^{1/2}}{\bar{F}(s)} = (1+o(1))\sqrt{2(n-1)}\left[\int_0^s u\bar{F}(u)du\right]^{1/2} s^{-1}.$$

- For $\alpha < 2$

$$e(Z_{AK}(s)) \sim \frac{\sqrt{n^2(n-1)\ k_\alpha\ [\bar{F}(s)]^3}}{n\bar{F}(s)} = (1+o(1))\sqrt{(n-1)k_\alpha}\ [\bar{F}(s)]^{1/2}.$$

Note that in the case of $\alpha = 2$ and $\mathbb{E}([X_1]^2) = \infty$, the term $\int_0^s u\bar{F}(u)du$ is a slowly varying function which is upper bounded by s^ϵ for all $\epsilon > 0$.

Improved Asmussen and Kroese's estimator

Asmussen and Kortschak (2012)[9] introduced a related estimators for $Z_{AK}(s)$ with faster rates of decay for the two cases: $\alpha > 2$ and $1 < \alpha < 2$. If $\alpha > 2$, by applying the Taylor expansion, the probability $z(s) = \mathbb{P}(S_n > s)$ can be written in term of $\bar{F}(s)$ and its derivatives:

$$\mathbb{P}(S_n > s) = n\bar{F}(s) + nf(s)\mathbb{E}(S_{n-1}) + nf'(s)\frac{\mathbb{E}([S_{n-1}]^2)}{2!} + \cdots$$

They suggested to use S_{n-1} as control variable and then resulting is the estimator

$$Z_{AKo^{(1)}}(s) = Z_{AK}(s) + n(\mathbb{E}(S_{n-1}) - S_{n-1})f(s).$$

The exact rates of decay of this estimator are given when the first derivative of f exists.

Theorem 1.14 (Asmussen and Kortschak, 2012)**.** *Assume the first derivative of f exists and is regularly varying at infinite: $f'(s) = -\alpha(\alpha-1)s^{-\alpha-2}l_1(s)$.*

- *If $\alpha > 4$ or more general $\mathbb{E}(X_1^4) < \infty$, then*

$$\mathbb{V}ar(Z_{AKo^{(1)}}(s)) \sim \frac{n^2}{4}\ \mathbb{V}ar([S_{n-1}]^2)[f'(s)]^2.$$

- *If* $\alpha = 4$ *and* $\mathbb{E}(X_1^4) = \infty$, *then*

$$\mathbb{V}ar(Z_{AKo^{(1)}}(s)) \sim n^2(n-1)[f'(s)]^2 \int_0^s u^3 \bar{F}(u)du.$$

- *If* $2 < \alpha < 4$, *then* $\mathbb{V}ar(Z_{AKo^{(1)}}(s)) \sim n^2(n-1)\ k'_\alpha\ [\bar{F}(s)]^3$ *where*

$$k'_\alpha \quad = \quad \alpha \int_0^\infty \left[((1-u)\vee u)^{-\alpha} - 1 - \alpha u\right]^2 u^{-\alpha-1}du.$$

In the case of $1 < \alpha < 2$, Asmussen and Kortschak suggested the use of an importance sampling method to improve $Z_{AK}(s)$ and proposed to consider the estimator $Z_{AKo^{(2)}}(s) = Z^{(b)}(s) + nZ^{(c)}(s)$ where

$$Z^{(b)}(s) = n(n-1)[\bar{F}(s/(2(n-1)))]^2 \mathbb{I}_{\{S_n > s, X_{n-1}\wedge X_n \geq M_{n-2}\}}$$
$$Z^{(c)}(s) = (\bar{F}(s-S_{n-1}) - \bar{F}(s))\mathbb{I}_{\{M_{n-1}\leq s/(2(n-1))\}}R + \bar{F}(s)\mathbb{P}\left(M_{n-1} \leq \frac{s}{2(n-1)}\right)$$

with $R = \prod_{i=1}^{n-1} f(X_i)/\tilde{f}(X_i)$ and $\tilde{f}$ is an importance sampling density of the form $\tilde{L}(s)/s^{\tilde{\alpha}}$ such that $\tilde{\alpha} < 2\alpha - 2$. The variance of $Z_{AKo^{(2)}}(s)$ is equivalent to $O([f(s)]^2)$ when $s \to \infty$.
Thus, the study of Asmussen and Kortschak has improved the estimator $Z_{AK}(s)$ for all the case where the tail index $\alpha > 1$. The relative error is calculated as follows:

- For $1 < \alpha < 2$, then

$$e(Z_{AKo^{(2)}}(s)) \sim O(f(s))/(n\bar{F}(s)) = O(s^{-1}).$$

- For $2 < \alpha < 4$, then

$$e(Z_{AKo^{(1)}}(s)) \sim \frac{\left[n^2(n-1)\ k_\alpha\ [\bar{F}(s)]^3\right]^{1/2}}{n\bar{F}(s)} = (1+o(1))\sqrt{(n-1)\ k_\alpha}\ [\bar{F}(s)]^{1/2}.$$

- For $\alpha > 4$, then

$$e(Z_{AKo^{(1)}}(s)) \sim \frac{\left[\frac{n^2}{4}\ \mathbb{V}ar([S_{n-1}]^2)\ [f'(s)]^2\right]^{1/2}}{n\bar{F}(s)} = (1+o(1))\frac{\alpha(\alpha-1)}{2}\ \mathbb{S}d([S_{n-1}]^2)\ s^{-2}.$$

Some importance sampling estimators for the sums of i.i.d. regularly varying random variables

Importance sampling is a variance reduction technique that can be used in the Monte Carlo simulation. The idea behind importance sampling is that certain values of the random variables in a simulation have more impact on the parameter being estimated than others. If these values are emphasized by sampling more frequently, then the estimator variance can be reduced. Mathematically, the importance sampling is based on the Radon-Nikodym derivative. If $f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$ is the density of vector $\mathbf{X} = (X_1, X_2, \cdots, X_n)$, the probability $z(s) = P(S_n > s)$ will be simulated by a sample $\mathbf{X}^* = (X_1^*, X_2^*, \cdots, X_n^*)$ where

$$\begin{aligned} P(S_n > s) &= \mathbb{E}\big[\mathbb{I}_{\{X_1+X_2+\cdots+X_n>s\}}\big] = \int^{\mathbb{R}^n} \mathbb{I}_{\{x_1+x_2+\cdots+x_n>s\}} f(\mathbf{x}) d\mathbf{x} \\ &= \int^{\mathbb{R}^n} \mathbb{I}_{\{x_1+x_2+\cdots+x_n>s\}} \frac{f(\mathbf{x})}{f^*(\mathbf{x})} f^*(\mathbf{x}) d\mathbf{x} = \mathbb{E}\left(\mathbb{I}_{\{X_1^*+X_2^*+\cdots+X_n^*>s\}} \frac{f(\mathbf{X}^*)}{f^*(\mathbf{X}^*)} \right), \end{aligned}$$

where $f^*(\mathbf{x}^*)$ is the density function of $\mathbf{X}^*$. To construct an importance sampling estimator, we pay attention to find a biased density f^* such that the variance under the importance sampling estimator is less than the variance of crude Monte Carlo method. Asmussen and Kroese (2006)[10] introduced the first importance sampling estimator for $z(s)$ where the importance sample was chosen as: $X_1, \cdots, X_{n-1}$ which their densities remain the same while X_n^* is a Lomax random variable of parameter $(\alpha, 1)$ with density f_n^* i.e. $f_n^*(x) = \alpha/(1+x)^{1+\alpha}$.

$$\begin{aligned} \mathbb{P}(S_n > s) &= n\mathbb{P}(S_n > s, X_n = M_n) = n\ \mathbb{E}\left(\mathbb{I}_{\{S_n>s, X_n=M_n\}}\right) \\ &= n\ \mathbb{E}\left(\frac{f_n^*(X_n^*)}{f_n(X_n^*)} \mathbb{I}_{\{X_1+\cdots+X_{n-1}+X_n^*>s, X_n^*=M_n\}} \right). \end{aligned}$$

This formula results in an importance sampling estimator for $z(s)$, called $Z_1^{IS}(s)$:

$$Z_1^{IS}(s) = \frac{f^*(X_n^*)}{f(X_n^*)} \mathbb{I}_{\{X_1+\cdots+X_{n-1}+X_n^*>s, X_n^*=M_n\}}.$$

Asmussen and Kroese proved that this estimator is logarithmically efficient. Another importance sampling estimator for $z(s)$ is introduced by Dupuis et al. (2007)[23]. The sample was constructed in such a way that the estimator has a bounded relative error. Before going into details of this importance sampling method, we define the notation $X^{[a,b]}$ for the conditional random variable $X|X \in (a, b]$. If the density of X is $f_X(x)$ then the density of $X^{[a,b]}$ is

$$f_{X^{[a,b]}}(x) = f_X(x)/\mathbb{P}(X \in (a, b])\mathbb{I}_{\{a<x\leq b\}}.$$

The i^{th} variable of the importance sample X_i^* is created by mixing the original density $f_i = f$ with the density of conditional random variable $X_i^{[c(s-S_{i-1}^*),\infty]}$ where c is a constant in $(0,1)$ and $S_{i-1}^* = X_1^* + \cdots + X_{i-1}^*$

- For $i = 1$, X_1^* has the density

$$f_{X_1^*}(x) = p_1 f(x) + q_1 f_{X_1^{[cs,\infty]}}(x).$$

- For $1 < i < n$, if $S_{i-1}^* > s$ then X_i^* has the density $f(x)$, otherwise

$$f_{X_i^*}(x) = p_i f(x) + q_i f_{X_i^{[c(s-S_{i-1}^*),\infty]}}(x).$$

- X_n^* has the density $f(x)$ if $S_{n-1}^* > s$, otherwise

$$f_{X_n^*}(x) = f_{X_n^{[c(s-S_{n-1}^*),\infty]}}(x)$$

where $p_1, p_2, \cdots, p_{n-1}$ are real numbers in $(0,1)$ and $q_i = 1 - p_i$ for $i = 1, 2, \cdots, n-1$. This estimator, called $Z_2^{IS}(s)$, has a bounded relative error (see Dupuis et al. (2007)[23]). Hult and Svensson (2012)[36] developed Dupuis et al. (2007) method and found the upper bound for its relative error as a function of c and p_i, q_i; $i = 1, 2, \cdots, n$.

$$e(Z_2^{IS}(s)) \leq \frac{\sqrt{\mathbb{E}([Z_2^{IS}(s)]^2)}}{z(s)} \leq \frac{1}{n}\left(\sum_{i=1}^{n-1} \frac{c^{-\alpha}}{q_i} \prod_{j=1}^{i-1} \frac{1}{p_j} + \sum_{i=1}^{n-1} \frac{1}{p_i}\right)^{1/2}$$

and the parameters $p_i, q_i; i = 1, 2, \cdots, n$ which minimize the upper bound are

$$p_i = \frac{(n-i-1)c^{-\alpha/2} + 1}{(n-i)c^{-\alpha/2} + 1} \; ; \; q_i = 1 - p_i.$$

At these points, the minimum value for the upper bound is $((n-1)c^{-\alpha/2} + 1)/n$.

Our results

In the third chapter, we propose some estimators with vanishing relative error for $z(s) = P(S_n > s)$ where S_n is the sum of n i.i.d. regularly random variables of tail index α. Our estimators are constructed by mixing the conditional Monte Carlo with importance sampling method and

achieve better performance compared to the existing estimators. First, $z(s)$ is decomposed into

$$\begin{aligned}\mathbb{P}(S_n > s) &= \mathbb{P}(S_n > s, M_n > s) + \mathbb{P}(S_n > s, M_n \leq s) \\ &= \mathbb{P}(M_n > s) + \mathbb{P}(S_n > s, M_n \leq s) \\ &= \mathbb{P}(M_n > s) + \mathbb{P}(M_n \leq s) \times \mathbb{P}(S_n > s | M_n \leq s).\end{aligned}$$

Note that $\mathbb{P}(M_n > s)$ and $\mathbb{P}(M_n < s)$ are determined, we focus on the probability $\mathbb{P}(S_n > s|M_n < s)$. With the notation $X^{[a,b]} = X|X \in (a, b]$, we have

$$\mathbb{P}(S_n > s|M_n < s) = \mathbb{P}(X_1^{[0,s]} + \cdots + X_n^{[0,s]} > s).$$

Instead of using the max M_n as the control variable as Asmussen and Kroese, we define a discrete random variable N_s which is equal to the number of $X_i^{[0,s]}$ such that $X_i^{[0,s]} > s/n$. By this definition, N_s is a binomial variable of parameter p_{N_s} which is calculated as follows:

$$p_{N_s} = (F(s) - F(s/n))/F(s) \sim (1 + o(1))(n^{\alpha-1} - 1)z(s).$$

Applying the law of total probability and notice that $\mathbb{P}(X_1^{[0,s]} + \cdots + X_n^{[0,s]} > s, N_s = 0) = 0$, we have

$$\begin{aligned}\mathbb{P}(X_1^{[0,s]} + \cdots + X_n^{[0,s]} > s) &= \mathbb{P}(N_s = 1) \times \mathbb{P}(X_1^{[0,s]} + \cdots + X_n^{[0,s]} > s|N_s = 1) \\ &+ \mathbb{P}(N_s \geq 2) \times \mathbb{P}(X_1^{[0,s]} + \cdots + X_n^{[0,s]} > s|N_s \geq 2).\end{aligned}$$

The probability $N_s \geq 2$ is determined by

$$\begin{aligned}\mathbb{P}(N_s \geq 2) &= 1 - (1 - p_{N_s})^n - np_{N_s}(1 - p_{N_s})^{n-1} \sim \frac{n(n-1)}{2}[p_{N_s}]^2 + O([p_{N_s}]^2) \\ &\sim (1 + o(1))\frac{n(n-1)}{2}(n^{\alpha-1} - 1/n)^2[z(s)]^2 + O([z(s)]^2).\end{aligned}$$

The rate of convergence of $\mathbb{P}(N_s \geq 2)$ is equivalent to $[z(s)]^2$ which implies that we must focus on $\mathbb{P}(X_1^{[0,s]} + \cdots + X_n^{[0,s]} > s|N_s = 1)$. Our estimators, denoted by $Z_{NR^{(1)}}(s)$ and $Z_{NR^{(2)}}(s)$, are presented in the third chapter as follows:

- Estimator $Z_{NR^{(1)}}(s)$ is developed under the assumption that the density of X_i is $I(\alpha)$ times differentiable where $I(\alpha)$ is the integer part of α. The relative error of $Z_{NR^{(1)}}(s)$ satisfies
  - If $\alpha - I(\alpha) < 1/2$, then $e(Z_{NR^{(1)}}(s)) \sim O(z(s))$.
  - If $\alpha - I(\alpha) > 1/2$, then $e(Z_{NR^{(1)}}(s)) \sim O(s^{-I(\alpha)-1/2})$.

- Estimator $Z_{NR^{(2)}}(s)$ is constructed with the relative error which satisfies $e(Z_{NR^{(2)}}(s)) \sim O(s^{-\alpha/(1+\alpha)})$ for all $\alpha > 0$ derived without any assumption of the density. This rate of decay is better than $Z_{AK}(s)$ in the case of $\alpha < 1$.

1.2.3 The sum of dependent regularly varying variables and simulation

In the second and the third chapter, we focused on the sums of regularly varying variables without any dependent assumption. However, in fact, dependences exist frequently between variables. In some cases, the results with dependent assumption completely differ from the results without dependent assumption. For example, the probability that the sum of n co-monotone Lomax variables with the same tail index α exceeds s will asymptotically decrease n^α times compared to that under the independent assumption. In this thesis, we are interested in the asymptotic behaviors of dependence which is so-called tail dependence.

Definition 1.15. *The tail dependent coefficient of the positive random variables X_1, X_2 with distribution functions F_1 and F_2 is defined by*

$$\lambda_{1,2} = \lim_{u \to 1} \mathbb{P}(F_1(X_1) > u | F_2(X_2) > u).$$

If $\lambda_{1,2} > 0$, we say X_1 and X_2 are tail dependent and if $\lambda_{1,2} = 0$, we say X_1 and X_2 are tail independent. Note that X_1 and X_2 may be strongly dependent but they can be tail independent, Gaussian vector is an example.

Asymptotic results for the sums of dependent regularly varying variables

Albrecher and Asmussen (2005) studied tail distribution of the sum of two dependent heavy tail random variables. They considered the positive exchangeable random variables X_1, X_2 of the same continuous marginal distribution function F and obtained results for asymptotic behavior of $\mathbb{P}(X_1 + X_2 > s)$.

Proposition 1.16 (Albrecher and Asmussen, 2005)**.** *If X_1 and X_2 are regularly varying random variables of the same distribution function $F \in \mathcal{RV}_\infty(-\alpha)$, with $S_2 = X_1 + X_2$, then the ratio $\mathbb{P}(S_2 > s)/\bar{F}(s)$ has the lower bound satisfying*

$$\liminf_{s \to \infty} \frac{\mathbb{P}(S_2 > s)}{\bar{F}(s)} \geq 2 - \lambda_{1,2}$$

and the upper bound satisfying

$$\limsup_{s\to\infty}\frac{\mathbb{P}(S_2>s)}{\bar{F}(s)}\leq\begin{cases}\left(\lambda_{1,2}^{1/(\alpha+1)}+(2-\lambda_{1,2})^{1/(\alpha+1)}\right)^{\alpha+1}, & 0\leq\lambda_{1,2}\leq\frac{2}{3}\\ 2^{\alpha}(2-\lambda_{1,2}) & ,\frac{2}{3}<\lambda_{1,2}\leq 1.\end{cases}$$

These results were developed by Albrecher et al. (2006)[2] in the case of the non-identical marginal distribution functions. If the survival distribution functions of X_1 and X_2 are regularly varying and satisfy $\lim_{s\to\infty}\frac{\bar{F}_2(s)}{\bar{F}_1(s)}=c$ and the coefficient $\hat{\lambda}_{1,2}$ is defined by

$$\hat{\lambda}_{1,2}=\lim_{s\to\infty}\mathbb{P}(X_2>s|X_1>s)$$

then we have the lower bound and the upper bound for the ratio $\frac{\mathbb{P}(S_2>s)}{\bar{F}_1(s)}$:

- the lower bound

$$\liminf_{s\to\infty}\frac{\mathbb{P}(S_2>s)}{\bar{F}_1(s)}\geq 1+c-\hat{\lambda}_{1,2}$$

- the upper bound

$$\limsup_{s\to\infty}\frac{\mathbb{P}(S_2>s)}{\bar{F}_1(s)}\leq\begin{cases}\left(\hat{\lambda}_{1,2}^{1/(\alpha+1)}+(1+c-2\hat{\lambda}_{1,2})^{1/(\alpha+1)}\right)^{\alpha+1}, & 0\leq\hat{\lambda}_{1,2}\leq\frac{1+c}{3}\\ 2^{\alpha}(1+c-\hat{\lambda}_{1,2}) & ,\frac{1+c}{3}<\hat{\lambda}_{1,2}\leq 1\end{cases}$$

The result of Proposition 1.16 is a special case when marginal distributions are identical: $c=1$ and $\hat{\lambda}_{1,2}=\lambda_{1,2}$. An important result for the sums of dependent regularly varying variables is the sums of asymptotically independent variables. Jensen and Mikosch (2006)[37] provided the tail behavior of the sum of n dependent random variables $X_1, X_2, \cdots, X_n$ where X_1 is regularly varying while the other may be regularly varying or may be not.

Theorem 1.17 (Jessen and Mikosch, 2006)**.** *Assume X_1 is regularly varying with index $\alpha_1\geq 0$ and survival distribution function $\bar{F}_1$. Assume $X_2,\cdots,X_n$ are random variables which satisfy*

$$\lim_{s\to\infty}\frac{\mathbb{P}(X_i>s)}{\bar{F}_1(s)}=c_i^{+}\ \textit{for all}\ i=2,\cdots,n$$

for some non-negative numbers c_i^{+} and

$$\lim_{s\to\infty}\frac{\mathbb{P}(X_i>s,X_j>s)}{\bar{F}_1(s)}=0\ \textit{for}\ i\neq j$$

then

$$\lim_{s\to\infty}\frac{\mathbb{P}(S_n>s)}{\bar{F}_1(s)}=1+c_2^+ +\cdots c_n^+.$$

In particular, if marginal variables X_i are non-negative independent regularly varying random variables, then

$$\mathbb{P}(S_n>s)\sim\mathbb{P}(X_1>s)+\mathbb{P}(X_2>s)+\cdots+\mathbb{P}(X_n>s).$$

The assumptions in Theorem 1.17 are strong. If there exist indices i such that c_i^+ are not vanishing, the condition $\lim_{s\to\infty}\frac{\mathbb{P}(X_i>s,X_j>s)}{\bar{F}_1(s)}=0$ for all $i\neq j$ makes sure that X_i are asymptotically independent of the others. If both c_i^+ and c_j^+ are vanishing i.e. both tails of X_i and X_j are dominated by the tail of X_1. Even if X_i and X_j are tail dependent, the dependence has no impact on the tail of S_n. Yuen and Yin (2012)[51] expanded Geluk and Tang (2009)[34] and proposed some asymptotic results for the sum of long tail random variables. These results can be applied for the sum of regularly varying marginal variables. Yuen and Yin suggested three assumptions for dependence structure:

- Assumption 1: Assume that $\hat{\lambda}_{i,j}=\lim_{x_i\wedge x_j\to\infty}\mathbb{P}(X_i>x_i|X_j>x_j)=0$ for all $i\neq j$. This is so-called asymptotic independent.
- Assumption 2: Assume that there exist positive constants x_0 and c_0 such that the inequality

  $$\mathbb{P}(X_i>x_i|X_j=x_j)\leq c_0\bar{F}_i(x_i)$$

  holds for all $i\neq j$ and $x_i\wedge x_j>x_0$.
- Assumption 3: Assume that the dependence between the variables can be presented by an absolutely continuous copula $C(u_1,\cdots,u_n)$ and there exist positive numbers m and M such that the copula density satisfies

  $$m\leq c(u_1,\cdots,u_n)\leq M$$

  for all $(u_1,\cdots,u_n)\in(0,1)^n$.

It can be shown that assumption 3 is a special case of assumption 2 while assumption 2 is a special case of assumption 1. Some related interesting discussions can be found in Geluk and Tang (2009). The main result of Yuen and Yi's paper is in the following theorem.

Theorem 1.18 (Yuen and Yi, 2012)**.** *If $X_1, \cdots, X_n$ are regularly varying random variables with distributions $F_1, \cdots, F_n$ respectively and $(X_1^*, \cdots, X_n^*)$ is an independent copy of $(X_1, \cdots, X_n)$ and has independent components then*

- *Under the assumption (i), we have*

$$\liminf_{s\to\infty} \frac{\mathbb{P}(S_n > s)}{\sum_{i=1}^{n} \bar{F}_i(s)} \geq 1.$$

- *Under the assumption (ii) and if additionally $\mathbb{P}(X_1^* + \cdots + X_n^* > s) \sim \sum_{i=1}^{n} \bar{F}_i(s)$, then*

$$\mathbb{P}(S_n > s) \sim \sum_{i=1}^{n} \bar{F}_i(s).$$

We can see that if $X_1, X_2, \cdots, X_n$ are dependent regularly varying variables, $X_1^*, X_2^*, \cdots, X_n^*$ are also regularly varying. According to Theorem 1.17, since $X_1^*, X_2^*, \cdots, X_n^*$ are independent, we have: $\mathbb{P}(X_1^* + \cdots + X_n^*) \sim \sum_{i=1}^{n} \bar{F}_i(s)$ i.e. the additional assumption of Theorem 1.18 is verified.
We can conclude that under (ii): $\mathbb{P}(S_n > s) \sim \sum_{i=1}^{n} \bar{F}_i(s)$.
The asymptotic results in Jessen and Mikosch (2006) or Yuen and Yi (2012) do not require any specific dependent model but it is hard to expand these works further. In the next section, we will introduce some studies where the dependence follows some parametric models.

Asymptotic results for the sums of regularly varying variables and Archimedean copulas

The notion of copula was introduced in Sklar (1959)[48] to decompose an n-dimensional distribution function $\mathbf{F}$ into two parts, the marginal distribution functions F_i and copula C, describing the dependence part of the distribution. See A.2 for more details about copulas.
Copulas have an important property that they do not change after strictly increasing transformations of marginal variables, i.e copula of $(X_1, \cdots, X_n)$ and copula of $(g_1(X_1), \cdots, g_n(X_n))$ are the same if all g_i are strictly increasing functions. This property allows us to study marginal distributions and dependence structure of multivariate distribution functions separately. This is why copulas are useful in defining new measures of dependence as concordance, tail dependence coefficient, etc...(see Appendix A.2)
Archimedean copula is a family of copulas indexed by a class of univariate functions, which was first introduced by Nelsen (1999)[44]. Construction of an Archimedean copula is based on

a decreasing function Φ, called Archimedean generator. For bivariate case, Nelsen found the condition for Φ to be an Archimedean generator.

Theorem 1.19 (Nelsen, 1999)**.** *Let Φ be a continuous, strictly decreasing function from $[0,\infty)$ to $[0,1]$ such that $\Phi(0)=1$, and let $\Phi^{\leftarrow}$ be the inverse of Φ. Let C be the function from $[0,1]^2$ to $[0,1]$ given by*

$$C(u,v)=\Phi\left(\Phi^{\leftarrow}(u)+\Phi^{\leftarrow}(v)\right)$$

then C is a copula if and only if Φ is convex.

An n-dimensional Archimedean copula is defined by

$$C(u_1,u_2,\cdots,u_n)=\Phi\left(\Phi^{\leftarrow}(u_1)+\Phi^{\leftarrow}(u_2)+\cdots+\Phi^{\leftarrow}(u_n)\right).$$

McNeil (2009)[42] studied the conditions for Φ to be Archimedean generator of an n-dimensional Archimedean copula. The details of generator of n dimensional Archimedean will be discussed in Section 4.2.

In the fourth chapter, we are interested in simulation method to calculate the probability $\mathbb{P}(S_n > s)$, where X_i are regularly varying and the dependence is an Archimedean copula or the Archimedean survival copula. Mario V.Wuthrich (2003)[50] suggested that the limit $q_n(\alpha,\beta)=\frac{\mathbb{P}(S_n>s)}{\mathbb{P}(X_1>s)}$ exists under the conditions that the marginal survival distribution functions $\bar{F}_1,\bar{F}_2,\cdots,\bar{F}_n$ are identical and satisfy $\bar{F}_i\in\mathcal{RV}_\infty(-\alpha)$; moreover, the dependence is an Archimedean survival copula of generator Φ where $\Phi\in\mathcal{RV}_\infty(-\beta)$. Wuthrich also provided a numerical algorithm to calculate $q_n(\alpha,\beta)$. (see Section 4.1). Likewise, Sun and Li (2010)[49] studied the relation between dependence structure of Archimedean copulas and dependence structure of multivariate regularly varying vectors to achieve integral form for the limit $\frac{\mathbb{P}(S_n>s)}{\bar{F}_1(s)}$ in the cases of Archimedean copulas and Archimedean survival copulas.

Definition 1.20. *A random vector $\boldsymbol{X}=(X_1,X_2,\cdots,X_n)$ of multivariate distribution function $F(x_1,x_2,\cdots,x_n)$ is said to be multivariate regularly varying if there exists a Radon measure μ (i.e. the measure is finite on compact sets), on $\mathbb{R}^n/\{\boldsymbol{0}\}$ such that*

$$\lim_{s\to\infty}\frac{\mathbb{P}(\boldsymbol{X}\in sA)}{\mathbb{P}\left(||\boldsymbol{X}||>s\right)}=\mu(A)$$

for any compact set $A\in\mathbb{R}^n/\{\boldsymbol{0}\}$.

Note that if vector $\mathbf{X}$ is regularly varying then all the marginal distribution functions are regularly varying with the same tail index. The Radon measure μ depends on the choice of norm $||.||$. Under the assumption of multivariate regularly varying, Sun and Li introduced the

upper tail dependent function and the upper exponent function, denoted by $b^*(\mathbf{x})$ and $a^*(\mathbf{x})$, defined as follows

$$b_n^*(\mathbf{x}) = \lim_{\epsilon \to 0^+} \frac{\bar{C}(1-\epsilon x_1, \cdots, 1-\epsilon x_n)}{\epsilon} \quad \forall \mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^{n+}.$$
$$a_n^*(\mathbf{x}) = \sum_{1 \le i_1, \cdots, i_j \le n} (-1)^{j-1} b_j^*(x_{i_1}, \cdots, x_{i_j}).$$

The authors realized the relation between the Radon measure μ and the tail dependent functions.

Theorem 1.21 (Sun and Li, 2010)**.** *Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a non-negative regularly varying random vector with Radon measure μ, copula C and continuous marginal distributions $F_1, \cdots, F_n$. If the marginal survival distributions are tail equivalent (i.e. $\bar{F}_i(s) \sim \bar{F}_j(s)$ as $s \to \infty$ for any $i \neq j$) and regularly varying with tail index $\alpha > 0$, then the upper tail dependent function and the upper exponent function $b^*(\boldsymbol{x})$ and $a^*(\boldsymbol{x})$ exist. Moreover, they satisfy the following equations*

1. $b_n^*(\boldsymbol{w}) = \dfrac{\mu\left([\boldsymbol{w}^{-1/\alpha}, \infty]\right)}{\mu\left([1,\infty) \times \bar{\mathbb{R}}^{+n-1}\right)}$ *and* $a_n^* = \dfrac{\mu\left([\boldsymbol{0}, \boldsymbol{w}^{-1/\alpha}]^c\right)}{\mu\left(([0,1] \times \bar{\mathbb{R}}^{+n-1})^c\right)}$.
2. $\dfrac{\mu\left([\boldsymbol{w}, \infty]\right)}{\mu([\boldsymbol{0}, \boldsymbol{1}]^c)} = \dfrac{b_n^*(\boldsymbol{w}^{-\alpha})}{a_n^*(\boldsymbol{1})}$ *and* $\dfrac{\mu\left([\boldsymbol{0}, \boldsymbol{w}]^c\right)}{\mu([\boldsymbol{0}, \boldsymbol{1}]^c)} = \dfrac{a_n^*(\boldsymbol{w}^{-\alpha})}{a_n^*(\boldsymbol{1})}$.

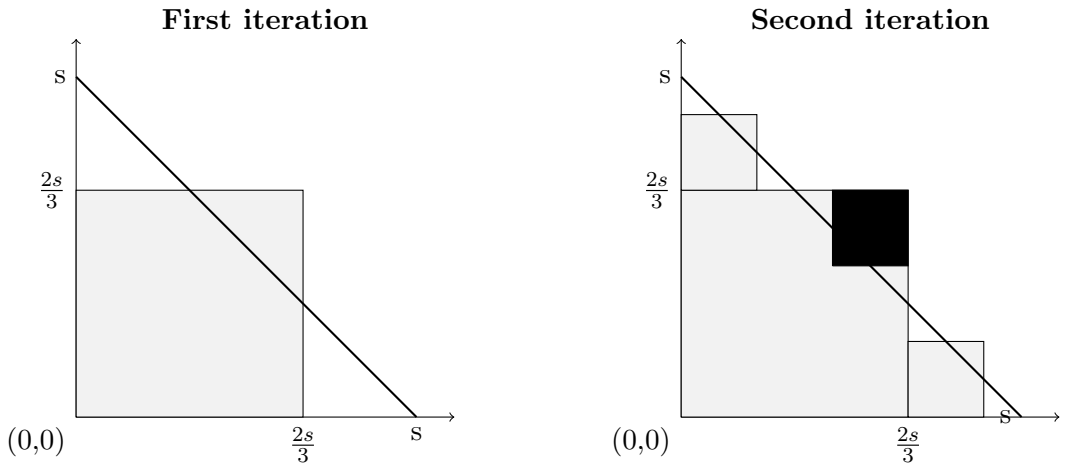
Through these results, the authors choose the norm $||.||$ as sum-norm. If the dependence between the variables is Archimedean copula of generator Φ satisfying $(1-\Phi) \in \mathcal{RV}_0(-\beta)$ and if the dependence is Archimedean survival copula of generator Φ satisfying $\Phi \in \mathcal{RV}_\infty(-\beta)$, the limits $\lim_{s\to\infty} \frac{\mathbb{P}(S_n > s)}{\mathbb{P}(X_1 > s)}$ exist and have integral forms. For more information, see Sun and Li (2010)[49], Equations 3.3, 3.4 or Section 4.1.

Numerical approximation for dependence sums

A well-known method to calculate distributions of the sum of dependence random variables is discretization of continuous distribution functions. For example, to approximate the continuous distribution function of vector **X**, the vector **X'** with discrete distribution function determined by

- Upper: $\mathbb{P}(\mathbf{X'} = \mathbf{k}h) = \mathbb{P}(\mathbf{X} \in (\mathbf{k}h, \mathbf{(k+1)}h])$.
- Lower: $\mathbb{P}(\mathbf{X'} = \ \mathbf{k}h) = \mathbb{P}(\mathbf{X} \in (\mathbf{(k-1)}h, \mathbf{k}h])$.
- Dispersion : $\mathbb{P}(\mathbf{X'} = \mathbf{k}h) = \mathbb{P}(\mathbf{X} \in (\mathbf{(k-1/2)}h, \mathbf{(k+1/2)} + h])$.

FIGURE 1.1: Azbenz 's numerical method



where $h > 0$ and $\mathbf{k} = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n$. The distribution function of sum $\mathbb{P}(S_n \leq s = Kh)$ where $K \in \mathbb{N}$ is approximated by

$$\mathbb{P}(\mathbf{X'} \leq Kh) = \sum^{\sum k_i \leq K} \mathbb{P}(\mathbf{X'} = \mathbf{k}h).$$

Note that results from the upper method are always higher than true probability while those from lower method are smaller. Generally, these approaches can be applied for any risk structure. In the context of calculating the sum of dependent regularly varying variables, they do not work since marginal distributions have heavy tails (due to the problem of choosing the bandwidth h) and because it is too hard to do the calculations in high dimension.

Arbenz et al. (2009)[6] presented a new numerical method to calculate the distribution of the sum by constructing a sequence of hyper-rectangles which the combination converges quickly to the hyper-triangle $\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n > s\}$. According to Figure 1.1, in each step, the probabilities in the gray squares are added, while the probability in the black square is subtracted and so on.

Cossette et al. (2014)[32] used the same idea of approximating hyper-triangle by the combination of hyper-rectangles to calculate lower bound (Figure 1.2) and upper bound (Figure 1.3) for the probability $1 - z(s) = \mathbb{P}(S_n \leq s)$. The authors mentioned that the computations are fast and converge to the exact value quickly with the assumption that the joint density function exists. Moreover, the approximation is deterministic, hence without error on the calculated values, is an advantage over simulation techniques.

As a whole, numerical methods by Arbenz and Cossette et al. can be applied for any positive marginal distributions, any dependence structure and any range of s. The inconvenience comes

FIGURE 1.2: Lower approximation of Cossette et al.

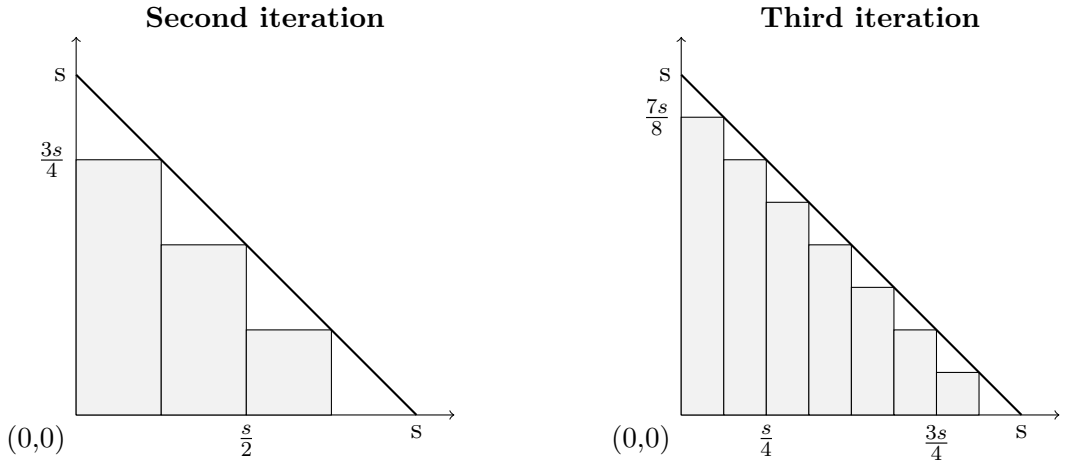
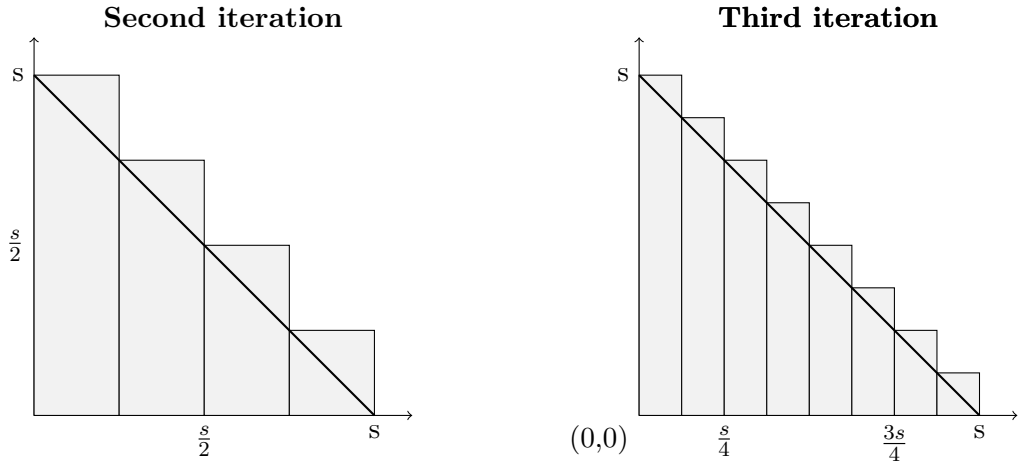


FIGURE 1.3: Upper approximation of Cossette et al.



from the calculation that takes a long time and lack precision in high dimension. Arbenz mentioned that the method has good performances when $n \leq 7$. (see Arbenz et al. (2009)[6]).

Calculation of dependent sums using simulation

Constructing an estimator $Z(s)$ of $z(s) = \mathbb{P}(S_n > s)$ under a dependent assumption is much more difficult under the case of independence. First, it is hard to simulate a dependent vector. Second, several classical methods are based on conditional distributions: if vector $\mathbf{X} = (X_1, \cdots, X_n)$ has the multivariate distribution function $F(x_1, \cdots, x_n)$ then the conditional distributions of

$X_i|(X_{i-1},\cdots,X_1)=(x_{i-1},\cdots,x_1)$ are calculated by

$$\begin{aligned} F_{X_i|X_{i-1},\cdots,X_1}(x_i) &= \mathbb{P}(X_i\le x_i|(X_{i-1},\cdots,X_1)=(x_{i-1},\cdots,x_1)) \\ &= \frac{\delta^{i-1}F(x_1,\cdots,x_{i-1},x_i,\infty,\cdots,\infty)}{\delta x_1\cdots\delta x_{i-1}}/f_{i-1}(x_{i-1},\cdots,x_1) \end{aligned}$$

where $f_{i-1}(x_{i-1},\cdots,x_1)$ is the density of $(X_{i-1},\cdots,X_1)$. The procedure of simulating $\mathbf{X}$ is then:

- Simulate $w_1,\cdots,w_n$ are i.i.d. $Uniform(0,1)$ random variables.
- Calculate $x_1=F_1^{\leftarrow}(w_1)$.
- For $i=2,3,\cdots,n$, calculate x_i by $x_i=F^{\leftarrow}_{X_i|X_{i-1},\cdots,X_1}(w_i)$.

However, almost all dependence structures do not have close forms for $F^{\leftarrow}_{X_i|X_{i-1},\cdots,X_1}(w_i)$ while numerical calculation of these n inverse functions will take a long time or impossible to do. In fact, dependence structures are simulated based on some special stochastic representations rather than the classical method. Beside the difficulties of simulating dependence structure, another problem of estimating $z(s)$ comes from the inefficiency of estimators. As we already discussed, when $z(s)$ is close to 0, the relative error of crude Monte Carlo estimator is equivalent to $[z(s)]^{-1/2}$ which is inefficient in rare event simulation.

There are some works relating to the calculations of tail distribution of dependent sums using simulations. Blanchet and Rojas-Nandayapa (2011)[17] proposed a conditional Monte Carlo estimator for $z(s)=\mathbb{P}(e_1^X+e_2^X+\cdots+e_n^X>s)$ where $\mathbf{X}=(X_1,X_2,\cdots,X_n)$ is an elliptic vector. Note that if $\mathbf{X}$ is an elliptic vector, then it has the stochastic representation

$$\mathbf{X}=\mu+R\mathbf{A}\Theta \quad i.e.\ for\ \ i=1,2,\cdots,n:\ \ X_i=\mu_i+R\langle\mathbf{A}[,i],\Theta\rangle$$

where $\mu=(\mu_1,\cdots,\mu_n)\in\mathbb{R}^n$; R is a random variable; $\mathbf{A}\mathbf{A}^T=\Sigma$ the covariance matrix where $\mathbf{A}[,i]$ is the i^{th} row of $\mathbf{A}$ and Θ is a uniform vector on unit sphere $\mathbb{S}_n^2=\{\theta_1,\cdots,\theta_n:\theta_1^2+\cdots+\theta_n^2=1\}$. Blanchet defined the function $G(r,\theta)$

$$G(r,\theta)=\sum_{i=1}^{n}\exp(\mu_i+r\langle\mathbf{A}[,i],\theta\rangle)$$

where $r\in\mathbb{R}$ and $\theta\in\mathbb{S}_n^2$. The probability $\mathbb{P}(e_1^X+e_2^X+\cdots+e_n^X>s)$ becomes $\mathbb{P}(G(R,\Theta)>s)$ which resulted in conditional Monte Carlo estimator

$$L(s,\Theta)=\mathbb{P}(G(R,\Theta)>s|\Theta).$$

With the assumption that random variable R has density f_R and survival distribution $\bar{F}_R$ satisfy

$$\lim_{s\to\infty} \frac{s f_R(s)}{[\bar{F}_R(s)]^{1-\epsilon}} = 0 \ \forall \ \epsilon > 0.$$

and function $G(.)$ satisfies assumptions (1),(2),(3) and (4) in 5^{th} page in Blanchet (2011), this estimator is logarithmically efficient. Kortschak and Hashorva (2013)[41] modified Asmussen and Kroese's method[10] to estimate tail distribution of the sum of log-normal variables and resulted in an estimator with vanishing relative error. As stated in Kortschak and Hashorva (2013) "the reason that Asmussen Kroese's estimator has a good asymptotic behavior in the asymptotically independent case is that heuristically when the sum is large then one element is large and all the others behave in a normal way". Applying this method for the asymptotically independent structure as log-normal vector, the performance of Kortschak and Hashorva's estimator is certainly better than that of Blanchet and Rojas-Nandayapa. Suppose that $\mathbf{X}$ is a log-normal vector, the probability $\mathbb{P}(\sum_{i=1}^{n} e^{X_i} > s)$ is developed by the Asmussen and Kroese's method:

$$\mathbb{P}(\sum_{i=1}^{n} e^{X_i} > s) = \left(\sum_{i=1}^{n} \mathbb{P}(e^{X_i} > s)\right) \times \sum_{j=1}^{n} \mathbb{P}(L = j) \ \frac{\mathbb{P}(\sum_{i=1}^{n} e^{X_i} > s, X_j = M_n)}{\mathbb{P}(e^{X_j} > s)}$$

where $M_n = \max(X_1, \cdots, X_n)$ and L is discrete random variable taking the value $j = 1, 2, \cdots, n$ with probability $\mathbb{P}(L = j) = \frac{\mathbb{P}(e^{X_j} > s)}{\sum_{i=1}^{n} \mathbb{P}(e^{X_i} > s)}$ respectively. Then, the estimator is proposed by

$$Z_{MAK}(s) = \left(\sum_{i=1}^{n} \mathbb{P}(e^{X_i} > s)\right) \times \sum_{j=1}^{n} \mathbb{I}_{\{L=j\}} \ \frac{Z_j(s)}{\mathbb{P}(e^{X_j} > s)}$$

where $Z_j = \mathbb{P}(\sum_{i=1}^{n} e^{X_i} > s, X_j = M_n | \mathbb{N}_{-j})$ with $\mathbb{N}_{-j} = (N_1, N_2, \cdots, N_{j-1}, N_{j+1}, \cdots, N_n)$ are $(n-1)$ standard normal random variables. For each j, the matrix $\mathbf{A}$ which $\mathbf{A}\mathbf{A}^T = \Sigma$ is chosen in such a way that the $\mathbf{A}[j,k] = 0$ for all $k \neq j$.

Our results

In the fourth chapter, we propose different approaches to simulate the probability $z(s) = \mathbb{P}(S_n > s)$ with the assumptions that marginal distributions are regularly varying and the dependence is an Archimedean copula or the Archimedean survival copula. The simulation has the advantage that it is applicable when marginal distributions are different. Moreover, the results via

simulation can be improved by increasing the number of replications. Our estimators are constructed based on the procedure of simulating a conditional Archimedean copula in Brechmann (2013)[19] and the stochastic representation of Archimedean copulas in McNeil (2009)[42].

Brechmann et al. provided an algorithm to simulate conditional vector $(X_1, \cdots, X_n | X_i = x_i)$ under the dependence structures of elliptic copulas, Archimedean copulas and vines copulas. In this section, we are interested in the type of conditional vector as $(X_1, \cdots, X_n | X_i \in A)$ where the dependence is an Archimdean copula or the Archimedean survival copula and A is an extreme region. As we reviewed, classical method of simulating dependent vector does not work because the inverse functions of conditional distributions have no close forms. Brechmann et al. introduced a procedure to simulate the vector $(X_1, \cdots, X_n | X_i \in A)$ using an intermediate random variable Z distribution function of which is called Kendall distribution. If copula of $\mathbf{U} = (U_1, \cdots, U_n)$ is C then Kendall distribution function is defined by $F_Z(t) = \mathbb{P}(C(\mathbf{U}) \leq t)$. Under Archimedean copulas, the inverse of conditional distributions of $U_i | U_{i-1}, \cdots, U_1$ has not got close forms but the inverses function of conditional distributions of $U_i | Z, U_{i-1}, \cdots, U_1$ do.

Archimedean copulas have a nice stochastic representation according to McNeil (2009) and it is useful to construct our estimators. If $\mathbf{U} = (U_1, \cdots, U_n)$ is an Archimedean copula of generator Φ then

$$(U_1, \cdots, U_n) = (\Phi(RW_1), \cdots, \Phi(RW_n))$$

where R is a positive random variable whose survival distribution function is calculated from Φ and its derivatives; $\mathbf{W} = (W_1, \cdots, W_n)$ is a uniform vector on unit simplex $\mathbb{S}_n = \{\mathbf{w} = (w_1, \cdots, w_n) : \sum_{i=1}^{n} w_i = 1\}$.

Our first estimator is developed from the procedure of simulating conditional vector under Archimedean copula of Brechmann et al. (2013). The probability $\mathbb{P}(S_n > s)$ is written

$$\mathbb{P}(S_n > s) = \mathbb{P}(M_n > s) + \mathbb{P}(S_n > s, s/n < M_n \leq s).$$

Note that the probability $\mathbb{P}(M_n > s)$ is determined by the marginal distributions and the copula function (see Section 4.3). We apply Asmussen and Kroese's method to obtain

$$\begin{aligned} \mathbb{P}(S_n > s, s/n < M_n \leq s) &= \sum_{i=1}^{n} \mathbb{P}(S_n > s, X_i = M_n, s/n < X_i \leq s) \\ &= \sum_{i=1}^{n} \mathbb{P}(X_i \in (s/n, s]) \, \mathbb{P}(S_n > s, X_i = M_n | s/n < X_i \leq s). \end{aligned}$$

Via this formula, we can propose the first estimator for $z(s)$, called $Z_{NR1}(s)$, where conditional probability $\mathbb{P}(S_n > s, X_i = M_n | s/n < X_i \leq s)$ is estimated by an indicator function. Because $\mathbb{P}(X_i \in (s/n, s])$ is determined and less than $z(s)$ for all i and indicator functions take the value 0 or 1, we can see that the estimator $Z_{NR1}(s)$ has bounded relative error. Note that this formula can be obtained for any dependence structure, this estimator can be applied for any type of dependence with the condition that we can simulate the vector $(X_1, \cdots, X_n | s/n < X_i \leq s)$ to result in indicator functions.

Our second estimator called $Z_{NR2}(s)$ is obtained from the stochastic representation of Archimedean copula of McNeil (2009)[42]. The estimation technique is similar to conditional Monte Carlo method used by Blanchet and Rojas-Nandayapa to estimate the sum of log-elliptic random variable. The conditional vector of our estimator is uniform vector on the unit simplex instead of uniform vector on the unit sphere. Suppose that vectors $\mathbf{X} = (X_1, \cdots, X_n)$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$ are of the same marginal distribution functions $F_1, \cdots, F_n$ while the dependence of $\mathbf{X}$ is an Archimedean copula and the dependence of $\mathbf{Y}$ is an Archimedean survival copula, then we have stochastic representations of $\mathbf{X}$ and $\mathbf{Y}$

$$(X_1, \cdots, X_n) \overset{d}{=} (F_1^{\leftarrow}(\Phi(RW_1)), \cdots, F_n^{\leftarrow}(\Phi(RW_n))).$$
$$(Y_1, \cdots, Y_n) \overset{d}{=} (\bar{F}_1^{\leftarrow}(\Phi(RW_1)), \cdots, \bar{F}_n^{\leftarrow}(\Phi(RW_n))).$$

The probability $\mathbb{P}(S_n > s, M_n \leq s)$ will be estimated conditionally on $\mathbf{W} \in \mathbb{S}_n$

$$Z_{NR2}^X(s) = \mathbb{P}(M_n^X > s) + \mathbb{P}\left(\sum_{i=1}^n F_i^{\leftarrow}(\Phi(RW_i)) > s, \max_{i=1,2,\cdots,n}\{F_i^{\leftarrow}(\Phi(RW_i))\} \leq s | \mathbf{W}\right)$$
$$Z_{NR2}^Y(s) = \mathbb{P}(M_n^Y > s) + \mathbb{P}\left(\sum_{i=1}^n \bar{F}_i^{\leftarrow}(\Phi(RW_i)) > s, \max_{i=1,2,\cdots,n}\{\bar{F}_i^{\leftarrow}(\Phi(RW_i))\} \leq s | \mathbf{W}\right).$$

Quality of this estimator will be discussed in details in Section 4.3

Our third estimator called $Z_{NR3}(s)$ is presented by mixing method of Asmussen and Kroese combined with techniques of Conditional Monte Carlo where conditional vector is $\mathbf{W} \in \mathbb{S}_n$. We have the probability $\mathbb{P}(S_n > s, M_n \leq s)$ as follows

$$\mathbb{P}(S_n > s, M_n \leq s) = P(S_n > s, M_{n-1} \leq \lambda s, M_n \leq s) + P(S_n > s, M_{n-1} > \lambda s, M_n \leq s)$$

for all $\lambda \in (0, 1/n)$. Asmussen and Kroese's method has good performances when the sum of variables is dominated by a single X_i while the others have normal values. That is why this method is efficient in estimating $\mathbb{P}(S_n > s, M_{n-1} \leq \lambda s, M_n \leq s)$. For the second term, $\mathbb{P}(S_n > s, M_{n-1} > \lambda s, M_n \leq s)$, there are at least two variables taking large values, we estimate

it conditionally on **W** and the result has a good asymptotic behavior (see Section 4.3). Although this estimator has better numerical performances than the first and the second ones (see Section 4.4) but its quality is not verified because Asmussen and Kroese's method is not efficient under strong dependence or tail dependence structures.

Our fourth estimator is developed from the probability $\mathbb{P}(S_n > s, M_n \leq s)$ which is written by

$$\mathbb{P}(S_n > s, M_n \leq s) = \mathbb{P}(S_n > s, \kappa s < M_n \leq s) + \mathbb{P}(S_n > s, M_n \leq \kappa s) \quad \textit{for} \quad \kappa \in (1/n, 1).$$

Simulation technique here is mixed by the methods in simulating $Z_{NR1}(s)$ with the method in simulating $Z_{NR2}(s)$. Indeed, the first term in the above equation is estimated similarly to $Z_{NR1}(s)$ since it can be written by

$$\begin{aligned}\mathbb{P}(S_n > s, \kappa s < M_n \leq s) &= \sum_{i=1}^{n} \mathbb{P}(S_n > s, \kappa s < X_i \leq s, X_i = M_n) \\ &= \sum_{i=1}^{n} \mathbb{P}(X_i \in (\kappa s, s])\mathbb{P}(S_n > s, X_i = M_n | \kappa s < X_i \leq s).\end{aligned}$$

And the second term is written by

$$\mathbb{P}(S_n > s, M_n \leq \kappa s) = \mathbb{P}\left(S_n > s, M_n \leq \kappa s, M_{n-1} > \frac{1-\kappa}{n-1} s\right)$$

which contains $M_{n-1} > \frac{1-\kappa}{n-1} \times s$, i.e this is the situation where there are at least two variables in the sum taking large values. Therefore, this probability can be estimated conditionally on **W** and has bounded relative error.

1.3 Intermediary proofs

Verifying the series of Albrecher, Hipp and Kortschak (2010) in the case of Pareto

Suppose that $\alpha \in \mathbb{R}^+/\mathbb{N}$, we calculate the sum of n i.i.d. Pareto random variables of parameter (α, β). With $k = [\alpha + 1]$, following the Theorem 1.8 in Albrecher, Hipp and Kortschak (2010) [1]: $\bar{G}(s) = a_k(s) + O\big(\bar{F}^{(k-1)}(s)\big)$ where

$$a_k(s) = a_1(s) + n\sum_{j=0}^{k-2}(-1)^j f^{(j)}(s)\frac{E\big[(X_1 + \cdots + X_{n-1})^{j+1}\big]}{(j+1)!}.$$

In the case of Paretos with parameter (α, β), we have

$$\begin{aligned}
&a_1(s) = n\beta^\alpha s^{-\alpha} \\
&f^{(j)}(s) = (-1)^j \beta^\alpha \alpha(\alpha+1)\cdots(\alpha+j)s^{-\alpha-j-1} \\
&E\big[(X_1+\cdots+X_{n-1})^{j+1}\big] = \sum_{j_1+\cdots j_l=j+1} \frac{(j+1)!}{(j_1)!\cdots(j_l)!}\prod_{r=1}^{l}\frac{\alpha\beta^{j_r}}{\alpha-j_r}.
\end{aligned}$$

Hence, the survival distribution $\bar{G}(s)$ can be written as

$$\bar{G}(s) = n\beta^\alpha\big(s^{-\alpha} + \sum_{j=1}^{k-1}\frac{\alpha(\alpha+1)\cdots(\alpha+j-1)}{j!}\sum_{\sum j_l=j}\frac{(j)!}{(j_1)!\cdots(j_l)!}\prod_{r=1}^{l}\frac{\alpha\beta^{j_r}}{\alpha-j_r}s^{-(\alpha+j)}\big) + O\big(s^{-(\alpha+k-1)}(s)\big).$$

Proof of Proposition 1.12

The density function of $X_{(n-1)}$ is bounded by

$$f_{X_{(n-1)}}(x) = n(n-1)\ [F(x)]^{n-2}\bar{F}(x)\ f(x) \leq n(n-1)\bar{F}(x)\ f(x)$$

The second moment of $Z_3(s)$ is evaluated separately over the regions $X_{(n-1)} \leq s/n$ and $X_{(n-1)} > s/n$. Note that when $X_{(n-1)} \leq s/n$ then $s - S_{(n-1)} \vee X_{(n-1)} = s - S_{(n-1)}$.

$$
\begin{aligned}
& \mathbb{E}([Z_3(s)]^2) \\
= \; & \mathbb{E}\left(\left[\frac{\bar{F}\big(s - S_{(n-1)}\big)}{\bar{F}(X_{(n-1)})}\right]^2, X_{(n-1)} \leq s/n\right) + \mathbb{E}\left(\left[\frac{\bar{F}\big(s - S_{(n-1)} U X_{(n-1)}\big)}{\bar{F}(X_{(n-1)})}\right]^2, s/n < X_{(n-1)}\right) \\
\leq \; & \mathbb{E}\left(\left[\frac{\bar{F}\big(s - S_{(n-1)}\big)}{\bar{F}(X_{(n-1)})}\right]^2, X_{(n-1)} \leq s/n\right) + \mathbb{P}(s/n < X_{(n-1)}) \\
\leq \; & \mathbb{E}\left(\left[\frac{\bar{F}(s/n)}{\bar{F}(X_{(n-1)})}\right]^2, X_{(n-1)} \leq s/n\right) + \int_{s/n}^{\infty} n(n-1)\bar{F}(x)\; f(x)dx \\
\leq \; & [\bar{F}(s/n)]^2 \mathbb{E}\left(\left[\frac{1}{\bar{F}(X_{(n-1)})},\right]^2, X_{(n-1)} \leq s/n\right) + n(n-1) \int_0^{\bar{F}(s/n)} u du \\
\sim \; & n^{2+2\alpha}[z(s)]^2 \mathbb{E}\left(\left[\frac{1}{\bar{F}(X_{(n-1)})}\right]^2, X_{(n-1)} \leq s/n\right) + \frac{n^{3+2\alpha}(n-1)[z(s)]^2}{2} \\
\leq \; & n^{2+2\alpha}[z(s)]^2 \times \left(\frac{n(n-1)}{2} + \int_0^{s/n} \frac{\bar{F}(x)\; f(x)}{[\bar{F}(x)]^2} dx\right) \\
= \; & n^{2+2\alpha}[z(s)]^2 \times \left(\frac{n(n-1)}{2} - (2+2\alpha)\log(n) - \log(z(s))\right).
\end{aligned}
$$

It can be seen that the last value is asymptotically upper bounded by $[z(s)]^{2-\epsilon}$ for all $\epsilon > 0$, which implies that $Z_3(s)$ is a logarithmically efficient estimator.

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Chapter 2

Exact computation for the sum of independent Pareto variables

Asymptotic expansions for the tails of the sums of random variables with regularly varying tails are mainly derived in the case of identically distributed random variables or in the case of random variables with the same tail index. Moreover the higher-order terms are often given under the condition of existence of a moment of the distribution. In this paper we obtain infinite series expansions for convolutions of Pareto distributions with non-integer tail-indices. The Pareto random variables may have different tail-indices and different scale parameters. We naturally found the same constants for the first terms as given in the previous asymptotic expansions in the case of identically distributed random variables, but we are now able to give the next additional terms. Since our series expansion is not asymptotic, it may be also used to compute the values of quantiles of the distribution of the sum as well as other risk measures as the Tail Value at Risk. Examples of values are provided for the sum of at least five Pareto random variables and are compared to those determined via previous asymptotic expansions or via simulations.

2.1 Introduction

In recent years the class of heavy-tailed distributions has been becoming more and more important in the domain of risk management (see e.g. [10]). The distributions with regularly varying tails are an important subclass that is traditionally used by actuaries to model catastrophic losses in an insurance portfolio, by risk managers in banks to quantify operational risk (the risk of losses resulting from inadequate or failed internal processes, people and systems, or external

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events), by economists to model the income distribution of populations... A positive random variable X is said to have a regularly tails with index $\alpha > 0$, if

$$P(X > x) = x^{-\alpha} L(x), \quad x \geq 0,$$

where L is a slowly varying function at infinity. A central example is the Pareto distribution for which the slowly varying function is equal to a constant. Its survival distribution function is given by

$$P(X > x) = \left(\frac{x}{\beta}\right)^{-\alpha}, \quad x \geq \beta,$$

where $\beta > 0$ is called the scale parameter.

The tail probabilities of the sums of independent random variables with regularly varying tails are of primary interest for an efficient quantitative risk managment. Asymptotic expansions for these probabilities have been mainly studied in the case of identically distributed random variables. Albrecher, Hipp and Kortschak ([2]) provide an overview of available results on asymptotic approximations and propose a method to get an higher-order expansion under the assumption that the survival distribution function is smoothly varying with index α and order m (i.e. it is eventually m-times continuously differentiable and its m^{th} derivative is a regularly varying function with index $-m-\alpha$). Assume that $X_1, \ldots, X_n$ are independent and identically distributed random variables with a common survival distribution function $\bar{F}$ which is smoothly varying with index $\alpha > 0$ and order $\lceil \alpha \rceil$. For $m = \lceil \alpha \rceil < \alpha + 1$, they prove that

$$\begin{aligned} P\left(X_1 + \ldots + X_n > s\right) &= n\bar{F}\left(s\right) + n \sum_{k=1}^{m-1} \frac{(-1)^{k-1}\, \mathbb{E}\left[\left(X_1 + \ldots + X_{n-1}\right)^k\right]}{k!} f^{(k-1)}\left(s\right) \quad (2.1) \\ &\quad - \binom{n}{2} \bar{F}\left(s\right)^2 \left(1 - 2\alpha\right) B\left(1 - \alpha, 1 - \alpha\right) + o(\bar{F}\left(s\right)^2) \end{aligned}$$

where $f^{(j)}\left(s\right)$ is the j^{th} derivative of the density probability function and B is the Beta function. An approximation is also given when $m = \alpha + 1$.

The case of the sum of not identically distributed random variables has been little studied. Geluk, Peng and de Vries ([4]) investigate the asymptotic behavior of the tail of the convolution of two regularly varying random variables with the same tail index and under a second order regular variation condition. Nadarajah and Kotz ([11]) provide an asymptotic expansion of the density function of the convolution of two Lomax distributions (essentially Pareto distributions that have been shifted so that their support begin at zero). Their expansion is based on the Appell function of the first kind and can not be used to get the survival distribution function in a closed form. Both previous approaches do not seem to be easily extended to more than two random variables. Barbe and McCormick ([4]) consider the sum of n regularly varying random

variables with the same tail index and such that the survival distribution functions have an asympotitic expansion in powers of s^{-1}. They obtain a class of distributions which is stable by convolution and give algebraic rules to compute the coefficients of the asymptotic expansion. This expansion is however limited to the terms for which the moments of the random variables exist.

In this paper, we decide to focus on the case of the sum of Pareto random variables. Although the Pareto distribution is simple, the distribution of the sum of two or more independent Pareto random variables is not less simple than the distributions of the sums of other regularly varying random variables. In particular very few explicit analytical expressions of the convolutions are known. In [8], Hagstroem provides exact results for the sum of two and three random variables when $\alpha = \beta = 1$. In [6], Brennan, Reed and Sollfrey derive series expansions of the probability density functions for the case of the sum of two or three Pareto random variables with the same scale parameter $\beta = 1$ and non-integer tail-indices. In [5], Blum obtains a series expansion of the probability density function of the sum of n independent and identically distributed Pareto random variables with $\beta = 1$ and $0 < \alpha < 2$ ($\alpha \neq 1$). More recently Ramsay provides integral representations of the probability density functions of the sums of n independent and identically distributed Pareto random variables for the case where $\beta > 0$ and (i) α is a positive integer (see [13]) or (ii) α is not a positive integer (see [14]). A related work (Albrecher and Kortschak, [3]) also introduces an integral representation for the ruin probability with Lomax claims.

The main result of the paper (Theorem 1) provides an infinite series expansion in the case of Pareto random variables with non-integer tail-indices $X_1(\alpha_1, \beta_1), \ldots, X_n(\alpha_n, \beta_n)$. The Pareto random variables may have different tail-indices and different scale parameters. As a by-product we derive the series expansion for the sum of two independent infinite Hall-Pareto random variables. We also provide new analytical expressions for the probability survival functions of sums of two independent Pareto random variables with integer tail-indices.

We naturally found the same constants for the first terms as given in the previous asymptotic expansions in the case of identically distributed random variables, but we are now able to give the next additional terms. Since our series expansion is not asymptotic, it may be also used to compute the values of quantiles of the distribution of the sum as well as other risk measures as the Tail Value at Risk. Examples of values are provided for the sum of at least five Pareto random variables and are compared to those determined via previous asymptotic expansions or via simulations.

This paper includes five additional sections. In the second section we state our main result about the series expansions for the probability survival functions of sums of independent Pareto random variables with non-integer tail-indices. The third section aims at presenting the method for proving the main result and provides several important intermediary results. The next section presents series expansions for the sum of two Pareto random variables with integer tail-indices.

In Section 5, numerical applications are given. We first discuss the error that is made when using a truncated series expansion and compare our approximation to the asymptotic approximation proposed by Albrecher, Hipp and Kortschak ([2]). Then our method is used for approximating the values of high quantiles and their corresponding Tail Value at Risk for the distribution of the sum of at least five Pareto random variables. These values are compared to those determined via simulations. We also give some comments on the way the probability survival functions of the sums of Pareto random variables with integer tail-indices can be approximated by series expansions with respective almost-integer tail-indices. All the proofs of the results of Sections 3 and 4 are gathered in the last section.

2.2 Main result

We first introduce some notation. We denote by Γ the Gamma function, i.e. $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ for $\alpha \in \mathbb{R}\backslash\mathbb{Z}^-$, and by B the Beta function given by $B(\alpha,\gamma) = \Gamma(\alpha)\Gamma(\gamma)/\Gamma(\alpha+\gamma)$ for $\alpha,\gamma \in \mathbb{R}\backslash\mathbb{Z}^-, \alpha+\gamma \in \mathbb{R}\backslash\mathbb{Z}^-$.

Let, for $\alpha \in \mathbb{R}\backslash\mathbb{Z}^-$ and $k \in \mathbb{N}^*$,

$$h(\alpha,k) = \frac{1}{kB(\alpha,k)}$$

and set $h(\alpha,0) = 1$.

The family of functions $(c_n)_{n\geq 1}$is defined in the following way: for $(\alpha_1,\alpha_2,\ldots,\alpha_n) \in (\mathbb{R}\backslash\mathbb{Z}^-)^n$, if there exit some different coefficients $j_1,\ldots,j_k \in \{1,\ldots,n\}$ with $1 \leq k \leq n$ such that $\sum_{l=1}^k \alpha_{j_l}$ is integer then $c_n(\alpha_1,\alpha_2,\ldots,\alpha_n) = 0$, else

$$c_n(\alpha_1,\alpha_2,\ldots,\alpha_n) = \frac{\Gamma(1-\alpha_1)\ldots\Gamma(1-\alpha_n)}{\Gamma(1-\sum_{i=1}^n \alpha_i)}.$$

The family of functions $(w_n)_{n\geq 1}$is defined for $((\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n),k) \in (\mathbb{R}_+\backslash\mathbb{N}\times\mathbb{R}_+)^n \times \mathbb{N}^*$ by

$$w_n((\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n),k) = \sum_{0\leq k_1,\ldots,k_n\leq k}^{k_1+\ldots+k_n=k} \frac{k!}{k_1!\ldots k_n!}\prod_{j=1}^n \frac{\alpha_j\beta_j^{k_j}}{\alpha_j - k_j}.$$

Let $\Theta_n = \{1,2,\ldots,n\}$ and $\Theta_{1,n},\Theta_{2,n},\ldots,\Theta_{l_n,n}$ be the $l_n = 2^n-2$ subsets of Θ_n different from $\varnothing$ and Θ_n. We denote by $\Theta_{i,n}^c = \Theta_n\backslash\Theta_{i,n}$ the relative complement of $\Theta_{i,n}$ in Θ_n, and for each subset $\Theta_{i,n} = \{i_1,i_2,\ldots,i_j\}$ of size $|\Theta_{i,n}| = j$ and $k \in \mathbb{N}$, we define for $((\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n),k) \in$

$(\mathbb{R}_+\backslash\mathbb{N}\times\mathbb{R}_+)^n\times\mathbb{N}^*$

$$\begin{aligned} c(\Theta_{i,n}) &= c_{|\Theta_{i,n}|}(\alpha_{i_1},\alpha_{i_2},\ldots,\alpha_{i_j}) \\ h(\Theta_{i,n},k) &= h\left(\sum_{p=1}^{j}\alpha_{i_p},k\right) \\ w(\Theta_{i,n},k) &= w_{|\Theta_{i,n}|}((\alpha_{i_1},\beta_{i_1}),(\alpha_{i_2},\beta_{i_2}),\ldots,(\alpha_{i_j},\beta_{i_j}),k). \end{aligned}$$

We are now able to give the series expansion for the sum of n independent Pareto random variables with non-integer tail-indices.

Theorem 2.1. *If $X_1(\alpha_1,\beta_1),\ldots,X_n(\alpha_n,\beta_n)$ are n independent Pareto random variables such that $\alpha_1,\ldots,\alpha_n\in\mathbb{R}_+\backslash\mathbb{N}$, then, for $s\geq\beta_1+\ldots+\beta_n$,*

$$\begin{aligned} &P\left(X_1+\ldots+X_n>s\right) \\ =\ &\sum_{i=1}^{l_n}\Big(\prod_{j\in\Theta_{i,n}}\beta_j^{\alpha_j}\Big)c(\Theta_{i,n})s^{-\sum_{j\in\Theta_{i,n}}\alpha_j}\left[\sum_{k=0}^{\infty}h(\Theta_{i,n},k)w(\Theta_{i,n}^c,k)s^{-k}\right]+\Big(\prod_{j=1}^{n}\beta_j^{\alpha_j}\Big)c(\Theta_n)s^{-\sum_{j=1}^{n}\alpha_j}. \end{aligned}$$

The survival distribution function of the sum may be rewritten in the following way: for $s\geq\beta_1+\ldots+\beta_n$,

$$\begin{aligned} &P\left(X_1+\ldots+X_n>s\right) \\ =\ &\sum_{i=1}^{l_n}\Big(\prod_{j\in\Theta_{i,n}}P\left(X_j>s\right)\Big)c(\Theta_{i,n})\left[\sum_{k=0}^{\infty}h(\Theta_{i,n},k)w(\Theta_{i,n}^c,k)s^{-k}\right]+c(\Theta_n)\prod_{j=1}^{n}P\left(X_j>s\right). \end{aligned}$$

The series expansion is therefore build as the sum of products of the survival functions of the X_i with series expansions in power of s^{-1}. This is similar (at least for the first terms) to the asymptotic approximations given in Geluk, Peng and de Vries ([4]) for two positive regularly varying random variables.

In the case of identically distributed Pareto random variables $X_1(\alpha,\beta),\ldots,X_n(\alpha,\beta)$, Theorem 2.1 provides the following series expansion: for $s\geq n\beta$,

$$\begin{aligned} &P\left(X_1+\ldots+X_n>s\right) \\ =\ &\sum_{j=1}^{n-1}\binom{n}{j}\beta^{j\alpha}\frac{\Gamma(1-\alpha)^j}{\Gamma(1-j\alpha)}s^{-j\alpha}\sum_{k=0}^{\infty}\frac{1}{kB(j\alpha,k)}\left[\sum_{0\leq k_1,\ldots,k_{n-j}\leq k}^{k_1+\ldots+k_{n-j}=k}\frac{k!}{k_1!\ldots k_{n-j}!}\prod_{r=1}^{n-j}\frac{\alpha\beta^{k_r}}{\alpha-k_r}\right]s^{-k} \\ &+\beta^{n\alpha}\frac{\Gamma(1-\alpha)^n}{\Gamma(1-n\alpha)}s^{-n\alpha}. \end{aligned}$$

Let us compare the terms of our expansion with those of approximation (2.1) proposed by Albrecher, Hipp and Kortschak. We can choose $m=\lceil\alpha\rceil$ since the Pareto survival probability function is infinitely differentiable. The first $m+1$ terms of our expansion are obtained for $j=1$

with $k = 0, \ldots, m-1$ and $j = 2$ with $k = 0$. In the case of a Pareto random variable $X(\alpha, \beta)$, we have, for $k = 1, \ldots, m-1$,

$$\begin{aligned} f^{(k-1)}(s) &= (-1)^{k-1} \beta^{\alpha} \left[\prod_{i=0}^{k-1} (\alpha + i) \right] s^{-\alpha-k}, \quad s \geq \beta, \\ \mathbb{E}\left[(X_1 + \ldots + X_{n-1})^k \right] &= w_{n-1}((\alpha, \beta), \ldots, (\alpha, \beta), k). \end{aligned}$$

Simple calculations show that we get the same expansion for the first $m+1$ terms. The next terms of our expansion are obtained for $j = 1$ with $k = m$

$$\frac{n\beta^{\alpha}}{mB(\alpha, m)} \left[\sum_{0 \leq k_1, \ldots, k_{n-1} \leq m}^{k_1 + \ldots + k_{n-1} = m} \frac{k!}{k_1! \ldots k_{n-1}!} \prod_{r=1}^{n-1} \frac{\alpha \beta^{k_r}}{\alpha - k_r} \right] s^{-\alpha-m},$$

then $j = 2$ with $k = 1$, and so on.

2.3 Outline of the proof of the main result and intermediary results

The proof of Theorem 2.1 goes by induction. We first study the case of the sum of two standard Pareto random variables ("standard" means that the scale parameter of the Pareto random variables is equal to 1) and the case of two (non-standard) Pareto random variables. Then we introduce the class of the Hall infinite-Pareto distribution and give a calculation rule to derive the series expansion for the sum of two Hall infinite-Pareto random variables. The result is then used to derive the way to build the series expansion for the sum of three (or more) Pareto random variables. The proofs of the propositions given in this section have been postponed to Section 2.6.

We first give another expression for the function c_2. For $\alpha_1, \alpha_2 \in \mathbb{R}_+ \backslash \mathbb{N}$, $j, l \in \mathbb{N}$ and $s > 1$, let

$$\eta_{j,l}(s, \alpha_1, \alpha_2) = s^{-\alpha_1} \sum_{k=j}^{l} \frac{\alpha_2}{(\alpha_2 - k)} h(\alpha_1, k) s^{-k}.$$

Note that the sequences of partial sums, $\eta_{0,n}(s, \alpha_1, \alpha_2)$ and $\eta_{0,n}(s, \alpha_2, \alpha_1)$, are uniformly-convergent sequences in s on $(1, \infty)$ and therefore we may introduce, for $\alpha_1, \alpha_2 \in \mathbb{R}_+ \backslash \mathbb{N}$, $\beta_1, \beta_2 \in \mathbb{R}_+$ and $s > \max(\beta_1, \beta_2)$,

$$\varsigma(s, (\alpha_1, \beta_1), (\alpha_2, \beta_2)) = s^{\alpha_1 + \alpha_2} \left(\frac{1}{\beta_1^{\alpha_1} \beta_2^{\alpha_2}} - \eta_{0,\infty}\left(\frac{s}{\beta_2}, \alpha_1, \alpha_2 \right) - \eta_{0,\infty}\left(\frac{s}{\beta_1}, \alpha_2, \alpha_1 \right) \right).$$

Proposition 2.2. *For $\alpha_1, \alpha_2 \in \mathbb{R}_+\backslash\mathbb{N}$, $\beta_1, \beta_2 \in \mathbb{R}_+$,*

$$c_2(\alpha_1, \alpha_2) = \varsigma(2, (\alpha_1, 1), (\alpha_2, 1)) = \varsigma((\beta_1 + \beta_2), (\alpha_1, \beta_1), (\alpha_2, \beta_2)).$$

Note moreover that, for $\alpha_1, \alpha_2 \in \mathbb{R}_+\backslash\mathbb{N}$ and an integer $k \geq 1$,

$$c_2(\alpha_1 + k, \alpha_2) = \frac{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + 1)\dots(\alpha_1 + \alpha_2 + k - 1)}{\alpha_1(\alpha_1 + 1)\dots(\alpha_1 + k - 1)} c_2(\alpha_1, \alpha_2),$$

and that if $(\alpha_1 + \alpha_2)$ is an integer number, then $c_2(\alpha_1, \alpha_2) = 0$.

We now consider the case of the sum of two standard Pareto random variables.

Proposition 2.3. *If $X_1(\alpha_1, 1)$, $X_2(\alpha_2, 1)$ are two independent Pareto random variables such that $\alpha_1, \alpha_2 \in \mathbb{R}_+\backslash\mathbb{N}$, then, for $s \geq 2$,*

$$P\left(X_1 + X_2 > s\right) = \eta_{0,\infty}(s, \alpha_1, \alpha_2) + \eta_{0,\infty}(s, \alpha_2, \alpha_1) + c_2(\alpha_1, \alpha_2) s^{-(\alpha_1+\alpha_2)}.$$

We then study the case of the sum of two Pareto random variables having different scale parameters.

Proposition 2.4. *If $X_1(\alpha_1, \beta_1)$, $X_2(\alpha_2, \beta_2)$ are two independent Pareto random variables such that $\alpha_1, \alpha_2 \in \mathbb{R}_+\backslash\mathbb{N}$, then, for $s \geq \beta_1 + \beta_2$,*

$$\begin{aligned} P\left(X_1 + X_2 > s\right) &= \beta_1^{\alpha_1} \sum_{k=0}^{\infty} h(\alpha_1, k) w_1((\alpha_2, \beta_2), k) s^{-\alpha_1-k} + \beta_2^{\alpha_2} \sum_{k=0}^{\infty} h(\alpha_2, k) w_1((\alpha_1, \beta_1), k) s^{-\alpha_2-k} \\ &\quad + \beta_1^{\alpha_1} \beta_2^{\alpha_2} c_2(\alpha_1, \alpha_2) s^{-(\alpha_1+\alpha_2)}. \end{aligned}$$

Before giving the series expansion for the sum of three independent Pareto random variables, we study the convolution of two infinite Hall-Pareto random variables. We say that a random variable H belongs to the class of the infinite Hall-Pareto distributions, $H \sim Hall((\alpha_i, \gamma_i)_{i\geq 1})$, if there exist constants $0 < \alpha_1 < \alpha_2 < \dots$, $\gamma_i \in \mathbb{R}$ and $r > 0$ such that $\sum_{i=1}^{\infty} \gamma_i r^{-\alpha_i} = 1$ and

$$P(H > x) = \sum_{i=1}^{\infty} \gamma_i x^{-\alpha_i}$$

is a decreasing function on (r, ∞). This class has the property of stability by convolution like the class $\mathcal{P}^+_{\alpha,m}$ introduced by Barbe and McCormick [4].

Proposition 2.5. *Let $H_1 \sim Hall((\alpha_{i,1}, \gamma_{i,1})_{i\geq 1})$ and $H_2 \sim Hall((\alpha_{i,2}, \gamma_{i,2})_{i\geq 1})$ two independent random variables. For $i \geq 1$ and $j \geq 1$, define $X_i(\alpha_{1,i}, r_1)$ and $Y_j(\alpha_{2,j}, r_2)$ as two independent*

Pareto random variables. Then, for $s \geq r_1 + r_2$,

$$P(H_1 + H_2 > s) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\gamma_{1,i}}{r_1^{\alpha_{1,i}}} \frac{\gamma_{2,j}}{r_2^{\alpha_{2,j}}} P\left(X_i + Y_j > s\right).$$

This result is helpful to construct the series expansion for the sum of 3 independent Pareto variables because the sum of 2 independent Pareto variables has an infinite Hall-Pareto distribution.

Proposition 2.6. *If* $X_1(\alpha_1, \beta_1)$, $X_2(\alpha_2, \beta_2)$ *and* $X_3(\alpha_3, \beta_3)$ *are three independent Pareto random variables such that* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_+ \backslash \mathbb{N}$, *then, for* $s \geq \beta_1 + \beta_2 + \beta_3$,

$$\begin{aligned} & P\left(X_1 + \ldots + X_n > s\right) \\ = & \sum_{i=1}^{l_3} \Big(\prod_{j \in \Theta_{i,3}} \beta_j^{\alpha_j} \Big) c(\Theta_{i,3}) s^{-\sum_{j \in \Theta_{i,3}} \alpha_j} \left[\sum_{k=0}^{\infty} h(\Theta_{i,3}, k) w(\Theta_{i,3}^c, k) s^{-k} \right] + \Big(\prod_{j=1}^{3} \beta_j^{\alpha_j} \Big) c(\Theta_3) s^{-\sum_{j=1}^{3} \alpha_j}. \end{aligned}$$

The end of the proof of Theorem 2.1 goes now by induction. Consider the case of the sum of $n+1$ random variables assuming that the series expansion is true for the sum of n random variables. Then it suffices to follow the same lines as that given in the proof of Proposition 2.6 to conclude that the series expansion is also true for the sum of $n+1$ random variables. In order to shorten the paper, we do not give more details about the end of the proof, but they are, however, available upon request.

2.4 Some series expansions in the case of integer tail-indices

It is well-known that asymptotic expansions differ when the tail indices of the regularly varying distributions are integer (see e.g. [2]). It is easily seen that it is also the case for the Pareto distributions since the functions w_n are not defined for some values of k if one of the tail-indices is integer. We were not able to give a general result for the sum of n Pareto random variables with integer tail-indices, but we provide at least new results in the case where $n = 2$. The proofs of the propositions given in this section have also been postponed to Section 2.6.

We first consider the case of the sum of two standard Pareto random variables with integer tail-indices. We define the family of functions $\left(Q_j(\cdot)\right)_{j \geq 0}$ on $(0,1)$ by $Q_0(u) = -\ln(1-u)$ and the recurrence relation $Q_{j+1}(u) = uQ_j'(u)$ for $j \geq 0$. Note that the functions may also be defined as series expansions, $Q_j(u) = \sum_{k=1}^{\infty} k^{j-1} u^k$. Moreover, for $m_2 \in \mathbb{N}$ and $x \in \mathbb{R}$, let $\Phi_{0,m_2}(x) = 1$ and, for $m_1 \in \mathbb{N}^*$, let

$$\Phi_{m_1,m_2}(x) = (x + m_2 + 1)...(x + m_1 + m_2).$$

Then define the triangular array of constants $(\varphi_{j,m_1,m_2})_{j=0,\ldots,m_1}$ by

$$\Phi_{m_1,m_2}(x) = \varphi_{0,m_1,m_2} + \varphi_{1,m_1,m_2}x + \ldots + \varphi_{m_1,m_1,m_2}x^{m_1}.$$

Proposition 2.7. *If $X_1(m_1,1)$, $X_2(m_2,1)$ are two independent Pareto random variables such that $m_1, m_2 \in \mathbb{N}^*$, then, for $s \geq 2$,*

$$\begin{aligned} P(X_1+X_2>s) &= \eta_{0,m_2-1}(s,m_1,m_2) + \eta_{0,m_1-1}(s,m_2,m_1) + \frac{2}{B(m_1,m_2)} s^{-(m_1+m_2)} \ln\left(\frac{s}{2}\right) \\ &\quad - s^{-(m_1+m_2)} \frac{m_2}{(m_1-1)!} \sum_{j=0}^{m_1-1} \varphi_{j,m_1-1,m_2} \left(Q_j\left(s^{-1}\right) - Q_j(0.5)\right) \\ &\quad - s^{-(m_1+m_2)} \frac{m_1}{(m_2-1)!} \sum_{j=0}^{m_2-1} \varphi_{j,m_2-1,m_1} \left(Q_j\left(s^{-1}\right) - Q_j(0.5)\right) \\ &\quad + \left(1 - \eta_{0,m_2-1}(2,m_1,m_2) - \eta_{0,m_1-1}(2,m_2,m_1)\right) \left(\frac{s}{2}\right)^{-(m_2+m_1)}. \end{aligned}$$

It is also possible to characterize the survival distribution function of the sum of two standard Pareto random variables with integer tail-indices in the following way.

Proposition 2.8. *If $X_1(m_1,1)$, $X_2(m_2,1)$ are two independent Pareto random variables such that $m_1, m_2 \in \mathbb{N}^*$ and $m_1 \geq m_2$, then, for $s \geq 2$,*

$$\begin{aligned} P(X_1+X_2>s) &= \frac{1}{s(s-1)^{m_1-1}} + \frac{1}{s(s-1)^{m_2-1}} \\ &+ \sum_{i=1}^{m_2-1} \frac{(m_1+m_2-1)(m_1+m_2-2)\ldots(m_1+m_2-2i)}{(m_1-1)(m_1-2)\ldots(m_1-i)(m_2-1)(m_2-2)\ldots(m_2-i)} s^{-2i} \\ &\times \left[\frac{1}{(s-1)^{m_1-i}}\left(\frac{m_1-i}{m_1+m_2-2i} - \frac{1}{s}\right) + \frac{1}{(s-1)^{m_2-i}}\left(\frac{m_2-i}{m_1+m_2-2i} - \frac{1}{s}\right)\right] \\ &+ \frac{(m_1+m_2-1)(m_1+m_2-2)\ldots(m_1-m_2+1)}{(m_1-1)(m_1-2)\ldots(m_1-m_2+1)(m_2-1)!} s^{-(m_1+m_2)} f(s, m_1-m_2+1, 1) \end{aligned}$$

where, for $j \in \mathbb{N}$,

$$f(s,j+1,1) = \sum_{k=1}^{j} \frac{s^k}{k}\left(1 - \frac{1}{(s-1)^k}\right) + 2\log(s-1).$$

For examples, if $m_1 = m_2 = 1$, we get

$$P(X_1+X_2>s) = 2s^{-1} + 2s^{-2}\log(s-1), \qquad s \geq 2,$$

if $m_1 = m_2 = 2$, we get

$$P(X_1+X_2>s) = 2s^{-2} + 8s^{-3} - 4s^{-3}/(s-1) + 12s^{-4}\log(s-1), \qquad s \geq 2,$$

if $m_1 = 4, m_2 = 3$ we get

$$\begin{aligned} & P(X_1 + X_2 > s) \\ = \; & \frac{1}{s(s-1)^3} + \frac{1}{s(s-1)^2} + \frac{5}{s^2}\left[\frac{1}{(s-1)^3}\left(\frac{3}{5} - \frac{1}{s}\right) + \frac{1}{(s-1)^2}\left(\frac{2}{5} - \frac{1}{s}\right)\right] \\ & + \frac{30}{s^4}\left[\frac{1}{(s-1)^2}\left(\frac{2}{3} - \frac{1}{s}\right) + \frac{1}{(s-1)}\left(\frac{1}{3} - \frac{1}{s}\right)\right] + \frac{60}{s^7}\left[s(1 - \frac{1}{s-1}) + 2\log(s-1)\right]. \end{aligned}$$

We are also able to give the exact expression of the survival distribution function of the sum of two Pareto random variables with integer tail-indices having different scale parameters.

Proposition 2.9. *If $X_1(m_1, \beta_1)$, $X_2(m_2, \beta_2)$ are two independent Pareto random variables such that $m_1, m_2 \in \mathbb{N}^*$, then, for $s \geq \beta_1 + \beta_2$,*

$$\begin{aligned} & P(X_1 + X_2 > s) \\ = \; & \left(\frac{\beta_1}{\beta_2}\right)^{\alpha_1} \eta_{0,m_2-1}\left(\frac{s}{\beta_2}, m_1, m_2\right) + \left(\frac{\beta_2}{\beta_1}\right)^{\alpha_2} \eta_{0,m_1-1}\left(\frac{s}{\beta_1}, m_2, m_1\right) \\ & + 2\beta_1^{m_1}\beta_2^{m_2} \frac{1}{B(m_1, m_2)} s^{-(m_1+m_2)} \ln\left(\frac{s}{\beta_1 + \beta_2}\right) \\ & - \left(\frac{\beta_1}{\beta_2}\right)^{\alpha_1} \left(\frac{s}{\beta_2}\right)^{-(m_1+m_2)} \frac{m_2}{(m_1-1)!} \sum_{j=0}^{m_1-1} \varphi_{j,m_1-1,m_2}\left(Q_j\left(\beta_2 s^{-1}\right) - Q_j\left(\beta_2/(\beta_1+\beta_2)\right)\right) \\ & - \left(\frac{\beta_2}{\beta_1}\right)^{\alpha_2} \left(\frac{s}{\beta_1}\right)^{-(m_1+m_2)} \frac{m_1}{(m_2-1)!} \sum_{j=0}^{m_2-1} \varphi_{j,m_2-1,m_1}\left(Q_j\left(\beta_1 s^{-1}\right) - Q_j\left(\beta_1/(\beta_1+\beta_2)\right)\right) \\ & + \left(1 - \left(\frac{\beta_1}{\beta_2}\right)^{\alpha_1} \eta_{0,m_2-1}\left(\frac{\beta_1+\beta_2}{\beta_2}, m_1, m_2\right) - \left(\frac{\beta_2}{\beta_1}\right)^{\alpha_2} \eta_{0,m_1-1}\left(\frac{\beta_1+\beta_2}{\beta_1}, m_2, m_1\right)\right) \\ & \times \left(\frac{s}{\beta_1+\beta_2}\right)^{-(m_2+m_1)}. \end{aligned}$$

If $m_1 \geq m_2$, then

$$\begin{aligned} P(X_1 + X_2 > s) \; = \; & \frac{\beta_1^{m_1}}{s(s-\beta_2)^{m_1-1}} + \frac{\beta_2^{m_2}}{s(s-\beta_1)^{m_2-1}} \\ & + \sum_{i=1}^{m_2-1} \frac{(m_1+m_2-1)(m_1+m_2-2)\ldots(m_1+m_2-2i)}{(m_1-1)\ldots(m_1-i)(m_2-1)\ldots(m_2-i)} \frac{\beta_1^i \beta_2^i}{s^{2i}} \\ & \times \left[\frac{\beta_1^{m_1-i}}{(s-\beta_2)^{m_1-i}}\left(\frac{m_1-i}{m_1+m_2-2i} - \frac{\beta_2}{s}\right) + \frac{\beta_2^{m_2-i}}{(s-\beta_1)^{m_2-i}}\left(\frac{m_2-i}{m_1+m_2-2i} - \frac{\beta_1}{s}\right)\right] \\ & + \frac{(m_1+m_2-1)\ldots(m_1-m_2+1)}{(m_1-1)\ldots(m_1-m_2+1)(m_2-1)!} \frac{\beta_1^{m_1}\beta_2^{m_2}}{s^{(m_1+m_2)}} g(s, m_1-m_2+1, 1; \beta_1, \beta_2) \end{aligned}$$

where, for $j \in \mathbb{N}$,

$$g(s, j+1, 1; \beta_1, \beta_2) = \sum_{k=1}^{j} \frac{s^k}{k}\left(\frac{1}{\beta_1^k} - \frac{1}{(s-\beta_2)^k}\right) + \log\left(\frac{s}{\beta_1} - 1\right) + \log\left(\frac{s}{\beta_2} - 1\right).$$

2.5 Numerical applications

Theorem 2.1 provides series expansions that are not asymptotic. They may be used to compute the values of tail probabilities for any level s, the values of quantiles as well as other risk measures.

In practice, the series expansion is truncated to compute the values of the tail probabilities. We first give an asymptotic upper bound for the error due to the truntation. Then we compare our approximations to those proposed by Albrecher, Hipp and Kortschak ([2]) for the case of the sum of five identically distributed Pareto random variables. We have also chosen to illustrate the interest of our approximation by considering two examples of computations of Values at Risk and of Tail Values at Risk. In insurance risk management, they are widely used to quantify the risk of loss on a specific portfolio. For a given portfolio, $X_1 + \ldots + X_n$, and a probability κ, VaR_κ is defined as the minimum threshold value such that the probability that loss on the portfolio exceeds this value is smaller than $1 - \kappa$

$$\mathrm{VaR}_\kappa(X_1 + \ldots + X_n) = \inf\{x \in \mathbb{R} : P\left(X_1 + \ldots + X_n > x\right) \leq 1 - \kappa\},$$

and TVaR_κ is defined as the expected value of the loss given that the loss is larger than VaR_κ

$$\mathrm{TVaR}_\kappa(X_1 + \ldots + X_n) = \mathbb{E}\left(X_1 + \ldots + X_n | X_1 + \ldots + X_n > VaR_\kappa(X_1 + \ldots + X_n)\right).$$

In the first example we consider the sum of five and ten independent Pareto random variables with non-integer tail-indices and in the second example the sum of five independent Hall-Pareto random variables with non-integer tail-indices. Finally we consider the sum of ten standard Pareto random variables with unit tail-indices and study the quality of approximation by series expansions with almost-integer indices.

Remark 2.10. From a numerical point of view, it is also possible to consider the inversion of the following Laplace transform

$$\mathcal{L}\left(\bar{F}_n\right)(t) = \int_0^\infty \exp\left(-ts\right) P\left(X_1 + \ldots + X_n > s\right) ds = \frac{1}{t}\left[1 - \prod_{i=1}^n \mathbb{E}\left[\exp\left(-tX_i\right)\right]\right]$$

where

$$\mathbb{E}\left[\exp\left(-tX_i\right)\right] = \alpha_i (t\beta_i)^{\alpha_i} \int_{t\beta_i}^\infty u^{-\alpha_i - 1} e^{-u} du = \alpha_i (t\beta_i)^{\alpha_i} \Gamma(-\alpha_i, t\beta_i)$$

with $\Gamma(-\alpha_i, t)$ the (upper) incomplete Gamma function. This function may be calculated by a series expansion

$$\Gamma(-\alpha_i, t) = \Gamma(-\alpha_i) - \sum_{k=0}^\infty (-1)^k \frac{t^{k-\alpha_i}}{(k - \alpha_i) k!}$$

(see formula 6.5.29 in [1]). Thus the Laplace transform $\mathcal{L}\left(\bar{F}_n\right)$ has a form of a product of series expansions (see also [12]). However it is well-known that the numerical inversion of such Laplace transforms needs multi-precision computations of several values of $\mathcal{L}\left(\bar{F}_n\right)(t)$ (especially as the α_i are close to integers). This multi-precision is not necessary for our method which is therefore easier to implement in order to compute approximated values if n is not too large (say 20).

Asymptotic bound for the error of the approximation of the truncated series expansion

Let us define a truncated series expansion by

$$\bar{F}_{n,K}(s) = \sum_{i=1}^{l_n} \left(\prod_{j \in \Theta_{i,n}} \beta_j^{\alpha_j} \right) c(\Theta_{i,n}) \sum_{k=0}^{K} h(\Theta_{i,n}, k) w(\Theta_{i,n}^c, k) s^{-\left(\sum_{j \in \Theta_{i,n}} \alpha_j + k\right)} + \left(\prod_{j \in \Theta_n} \beta_j^{\alpha_j} \right) c(\Theta_n) s^{-\left(\sum_{j=1}^{n} \alpha_j\right)}.$$

The absolute error between $\bar{F}_n(s) = P\left(X_1 + \ldots + X_n > s\right)$ and $\bar{F}_{n,K}(s)$ may be bounded when K is large in the following way.

Proposition 2.11. *Let $e_{n,K}(s) = |\bar{F}_n(s) - \bar{F}_{n,K}(s)|$. We have, for $s \geq \beta_1 + \ldots + \beta_n$ and for large K,*

$$e_{n,K}(s) \leq (1 + o(1)) \sum_{i=1}^{l_n} \left(\prod_{j \in \Theta_{i,n}} \beta_j^{\alpha_j} \right) \left| c(\Theta_{i,n}) h(\Theta_{i,n}, K+1) w(\Theta_{i,n}^c, K+1) \right| \frac{s^{-\left(\sum_{j \in \Theta_{i,n}} \alpha_j + K\right)}}{s - s_{i,n}} \tag{2.2}$$

where $s_{i,n} = \sum_{j \in \Theta_{i,n}^c} \beta_j$.

Comparisons between the approximations for identically distributed random variables

We consider the sum of five independent and identically distributed Pareto random variables with tail-index α equal to 0.3, 0.7, 1.3 and 1.7 and scale parameter β equal to one. We compare our approximation $\bar{F}_{5,K}(s)$ (K has been chosen equal to 10 for $\alpha = 0.3, 0.7$ and K has been chosen equal to 30 for $\alpha = 1.3, 1.7$) with the approximation proposed by Albrecher, Hipp and Kortschak ([2]), $\bar{F}_{AHK}(s)$. $\bar{e}_{5,K}(s)$ denotes the upper bound of the error due to the truncation and $\Delta(s)$ is the difference between both approximations.

We see that, for low levels of s, the approximation $\bar{F}_{AHK}(s)$ may be poor, but the difference between both approximations becomes rapidly small when s increases.

TABLE 2.1: Comparisons of the approximations of the tail probabilities for the sum of 5 id Pareto variables with tail index α equal to 0.3

s	$\bar{F}_{5,K}(s)$	$\bar{e}_{5,K}(s)$	$\bar{F}_{AHK}(s)$	$\Delta(s)$
10^2	0.81869414145	3.1×10^{-17}	0.776658443075	4.2×10^{-2}
10^3	0.51400403322	2.7×10^{-28}	0.5090718141648	4.9×10^{-3}
10^4	0.2858390017	1.4×10^{-36}	0.285237847498	6.0×10^{-4}
10^6	7.7345858×10^{-2}	4×10^{-54}	7.7336592×10^{-2}	9.2×10^{-6}
10^9	9.9460890×10^{-3}	5.2×10^{-83}	9.9460707×10^{-3}	1.8×10^{-8}
10^{12}	1.2554639×10^{-3}	6.6×10^{-111}	1.2554639×10^{-3}	3.6×10^{-11}

TABLE 2.2: Comparisons of the approximations of the tail probabilities for the sum of 5 id Pareto variables with tail index α equal to 0.7

s	$\bar{F}_{5,K}(s)$	$\bar{e}_{5,K}(s)$	$\bar{F}_{AHK}(s)$	$\Delta(s)$
50	0.37236442352	8.1×10^{-9}	0.4239059060	-5.2×10^{-2}
100	0.22258296637	1.9×10^{-11}	0.2371520319	-1.5×10^{-2}
500	6.7665654×10^{-2}	1.2×10^{-17}	$6.8522166394 \times 10^{-2}$	-8.5×10^{-4}
10^3	4.0973532×10^{-2}	5.1×10^{-25}	4.1233138×10^{-2}	-2.6×10^{-4}
10^4	7.9797232×10^{-3}	1.1×10^{-34}	7.9848479×10^{-3}	-5.1×10^{-6}
10^6	3.1557231×10^{-4}	4.0×10^{-54}	3.1557437×10^{-4}	-2.0×10^{-9}

TABLE 2.3: Comparisons of the approximations of the tail probabilities for the sum of 5 id Pareto variables with tail index α equal to 1.3

s	$\bar{F}_{5,K}(s)$	$\bar{e}_{5,K}(s)$	$\bar{F}_{AHK}(s)$	$\Delta(s)$
10	0.6694535572	8.1×10^{-15}	0.6117374149	5.8×10^{-2}
50	0.1031895699	2.6×10^{-29}	0.0935046258	9.7×10^{-3}
100	4.3920691×10^{-5}	4.9×10^{-36}	4.1762231×10^{-5}	2.2×10^{-3}

TABLE 2.4: Comparisons of the approximations of the tail probabilities for the sum of 5 id Pareto variables with tail index α equal to 1.7

s	$\bar{F}_{5,K}(s)$	$\bar{e}_{5,K}(s)$	$\bar{F}_{AHK}(s)$	$\Delta(s)$
10	0.4524003842	6.2×10^{-14}	0.3301369271	0.12
20	7.6124077×10^{-2}	3.4×10^{-23}	6.2276372×10^{-2}	1.4×10^{-2}
30	2.8414974×10^{-2}	1.3×10^{-28}	2.5462374×10^{-2}	2.9×10^{-3}

First example of computations of risk measures with our expansion formulas

We consider the sum of five independent Pareto random variables. The parameters are generated randomly such that the tail indices are non-integers, and the scale parameters are positive ($\alpha_1 = 0.563, \beta_1 = 2.242$), ($\alpha_2 = 1.453, \beta_2 = 1.456$), ($\alpha_3 = 3.324, \beta_3 = 4.345$), ($\alpha_4 = 1.655, \beta_4 = 1.234$) and ($\alpha_5 = 4.245, \beta_5 = 0.835$).

In the following tables we compare the approximated values to those determined via simulations as well as the time needed to get the values. K gives the number of terms in the series expansion that are used for the approximations. All the computations have been done with the free software

R on a computer with the following characteristics: Core(TM) 2 Duo, CPU T9300 @ 2.50GHz, 772MHz 1.00 GB of RAM.

TABLE 2.5: VaR of the sum of 5 variables calculated by simulation

Number of simulations	500000	1000000	5000000	10000000
$(Time)$	$(5s)$	$(7s)$	$(28s)$	$(1m14s)$
$VaR_{0.1}$	13.80361	13.80991	13.81115	13.81149
$VaR_{0.5}$	22.73633	22.71770	22.71211	22.71277
$VaR_{0.95}$	480.0961	481.5711	476.1103	479.8297
$VaR_{0.99}$	8120.633	8045.054	8013.650	8026.669
$VaR_{0.995}$	28711.87	27027.48	27461.51	27616.38

TABLE 2.6: VaR of the sum of 5 variables calculated by our method

K	20	30	40	50
$VaR_{0.1}$	13.80290	13.81327	13.81352	13.81340
$(Time)$	$(2s)$	$(3s)$	$(5s)$	$(7s)$
K	5	10	15	20
$VaR_{0.5}$	22.81832	22.71466	22.71460	22.71485
$(Time)$	$(1s)$	$(2s)$	$(2s)$	$(2s)$
K	2	3	4	5
$(Time)$	$(\leq 1s)$	$(\leq 1s)$	$(\leq 1s)$	$(\leq 1s)$
$VaR_{0.95}$	478.1181	478.4145	478.4219	478.4221
$VaR_{0.99}$	8019.968	8019.987	8019.987	8019.987
$VaR_{0.995}$	27419.94	27419.95	27419.95	27419.95

Table 5 provides the approximations determined via simulations and Table 6 the approximations derived with our method. For the low and intermediate levels of probability κ ($\kappa = 0.1$, resp. 0.5), our method needs expansions with at least 30, resp. 10, terms to have accurate approximations (with errors less than 10^{-2}). The computations are done in less than 3 seconds. The Monte-Carlo method needs 10^6 simulations and 7 seconds to have acceptables approximations. For the high levels of probability κ ($\kappa \geq 0.95$), our method converges very quickly (only 2 or 3 terms are needed) and the time of calculation is less than 1 second, whereas the results obtained by simulations are unstable even if 10^7 simulations are performed.

We now add five more independent Pareto variables with parameters $(\alpha_6 = 0.563, \beta_6 = 2.242)$, $(\alpha_7 = 1.453, \beta_7 = 1.456)$, $(\alpha_8 = 3.324, \beta_8 = 4.345)$, $(\alpha_9 = 3.121, \beta_9 = 1.234)$ and $(\alpha_{10} = 4.245, \beta_{10} = 0.835)$.

Table 7 provides the approximations determined via simulations and Table 8 the approximations derived with our method. Having five random variables or ten in the sum neither modifies a lot the accuracy of the approximations nor the time of calculation for the Monte-Carlo method. Of course our method needs more time than the previous since there are more terms to calculate, but for the same accuracy it provides approximations at least three times faster.

TABLE 2.7: VaR of the sum of 10 variables calculated by simulation

Number of simulations $(Time)$	500000 $(6s)$	1000000 $(11s)$	5000000 $(53s)$	10000000 $(1m48s)$
$VaR_{0.1}$	46.56980	46.55908	46.56425	46.56102
$VaR_{0.5}$	67.14518	67.09037	67.15230	67.15699
$VaR_{0.95}$	556.2743	554.6995	557.4301	560.2321
$VaR_{0.99}$	8176.780	8011.923	8021.472	8249.039
$VaR_{0.995}$	28520.88	26519.30	27235.69	28047.29

TABLE 2.8: VaR of the sum of 10 variables calculated by our method

K	20	25	30	35
$VaR_{0.1}$	46.56262	46.55977	46.56125	46.56171
$(Time)$	$(52s)$	$(1m6s)$	$(1m24s)$	$(1m46s)$
K	14	16	18	20
$VaR_{0.5}$	67.14200	67.14465	67.14526	67.14520
$(Time)$	$(41s)$	$(45s)$	$(49s)$	$(52s)$
K	2	4	6	8
$VaR_{0.95}$	552.3559	559.7354	559.8288	559.8295
$(Time)$	$(31s)$	$(33s)$	$(34s)$	$(35s)$
K	2	3	4	5
$(Time)$	$(31s)$	$(32s)$	$(33s)$	$(34s)$
$VaR_{0.99}$	8139.306	8139.839	8139.844	8139.844
$VaR_{0.995}$	27569.51	27569.67	27569.67	27569.67

Second example of computations of risk measures with our expansion formulas

We consider the sum of five independent Hall-Pareto random variables H_i, $i = 1, 2, 3, 4, 5$, whose probability survival functions are given by

$$\begin{aligned}
P(H_1 > x) &= \left(0.717x^{-1.364} + 1.470x^{-2.164} + 0.512x^{-2.434} + 1.944x^{-3.346}\right) \\
P(H_2 > x) &= \left(2.732x^{-1.734} + 1.951x^{-2.759} + 0.029x^{-3.677}\right) \\
P(H_3 > x) &= \left(4.318x^{-2.182} + 3.143x^{-2.555} - 1.039x^{-2.900} + 0.534x^{-3.124} + 4.343x^{-4.011}\right) \\
P(H_4 > x) &= \left(1.227x^{-2.346} + 6.431x^{-2.789} + 1.788x^{-3.973}\right) \\
P(H_5 > x) &= \left(5.242x^{-1.904} + 0.435x^{-2.614}\right)
\end{aligned}$$

where $(r_1 = 1.900, r_2 = 2.109, r_3 = 2.454, r_4 = 2.188, r_5 = 2.442)$. The parameters have been generated randomly such that the expectation of each random variable exists.

By the method of simulation, it takes about 150 seconds to have a sample of size 10^5 for each random variable. Hence, to obtain a sample of size 10^5 for the sum, it takes about 750 seconds. Following the above example, the size 10^5 is however not enough to have accurate

approximations of Value at Risks with high levels of probability, and therefore a bigger sample is needed. But the time of calculation could then be quite long: for example, it takes more than 2 hours to have a sample of size 10^6.

TABLE 2.9: VaR and TVaR of the sum of 5 Hall variables

Level	10%	50%	95%	99%	99.5%
VaR	14.02788	18.15875	37.58615	70.44447	97.13317
TVaR	22.6820	27.99025	69.80856	158.8083	236.2707

By our method, following Proposition 2.5, to obtain the distribution of $(H_1+H_2+H_3+H_4+H_5)$, we have to calculate $4\times3\times5\times3\times2 = 360$ times the distribution of sum of 5 independent Pareto variables, which takes 1934 seconds (32.5 minutes) for the low levels of probability ($K = 50$). For the high levels, the time of calculation is about 5 minutes. We also derive approximations for the Tail Value at Risks of the sum, whereas the method of simulation can not give any acceptable result for the Tail Values at Risk in general.

Approximations of the tail probabilities for the sums of Pareto random variables with integer tail-indices

We finally discuss how the probability survival functions of the sums of Pareto random variables with integer tail-indices can be approximated by series expansions with respective almost-integer indices.

First recall that, if $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$ are independent positive random variables such that $P(X_i \le x) \le P(Y_i \le x)$ for $i = 1,\dots,n$ and for any $x \ge 0$, then $P(X_1+\dots+X_n \le x) \le P(Y_1+\dots+Y_n \le x)$ for any $x \ge 0$.

Let $\Xi \subset \Theta_n$ and assume that, for $j \in \Xi$, $\alpha_j \in \mathbb{N}^*$. Let $\varepsilon > 0$ and define, for all $i \in \Xi$, Y_i^+ (resp. Y_i^-) as a random variable with the same distribution as $X_i(\alpha_i+\varepsilon,\beta_i)$ (resp. $X_i(\alpha_i-\varepsilon,\beta_i)$) and independent of the other random variables. We then derive the following bounds

$$P\left(\sum_{i\in\Xi} Y_i^+ + \sum_{i\in\Xi^c} X_i > s\right) \le P\left(\sum_{i=1}^n X_i > s\right) \le P\left(\sum_{i\in\Xi} Y_i^- + \sum_{i\in\Xi^c} X_i > s\right). \tag{2.3}$$

Note that, if ε is very small, multi-precision may be required in practice to compute the constants $c(\Theta_{i,n})$ and $w(\Theta_{i,n},k)$.

In this last example, we consider the sum of ten standard Pareto random variables with unit tail indices. We approximate the values of the survival probabilities by using Equation (2.3) with $\varepsilon = 10^{-10}$ and give the values of the bounds of the errors given by Equation (2.2) for these approximations and for several values of K (see Table 6).

TABLE 2.10: Numerical lower and upper bounds for $\bar{F}(s)$

s = 11	$\bar{F}_K^+$ (Lower bound)	$\bar{F}_K^-$ (Upper bound)
K = 130	0.999973048452137	1.000094202688167
error	0.000053902537869	0.008771801784979
K = 140	0.999999956830654	0.999999962600813
error	0.000000005922260	0.000001131249689
K = 150	0.999999948351894	0.999999948353228
error	0.000000000003187	0.000000032479338
K = 160	0.999999948348678	0.999999948348680
error	$5.92 * 10^{-16}$	0.000000000000014
K = 200	0.999999948348677	0.999999948348677
error	$4.05 * 10^{-27}$	$1.3 * 10^{-24}$
s = 50	$\bar{F}_K^+$ (Lower bound)	$\bar{F}_K^-$ (Upper bound)
K = 50	-57864569	76040232
error	1077334999	3325809546
K = 60	0.381768396092029	0.381771899873744
error	0.000038914282494	0.000072936507305
K = 70	0.381770141115882	0.381770141333152
error	$8.09 * 10^{-19}$	$1.85 * 10^{-18}$
K = 80	0.381770141115882	0.381770141333152
error	$5.94 * 10^{-32}$	$2.01 * 10^{-31}$
s = 1000	$\bar{F}_K^+$ (Lower bound)	$\bar{F}_K^-$ (Upper bound)
K = 30	0.110878105067172	0.110883997759744
error	77.1462491867784	78.7702652831532
K = 40	0.010649942642845	0.010649942657699
error	$7.32 * 10^{-28}$	$7.57 * 10^{-28}$
K = 50	0.010649942642845	0.010649942657699
error	$1.33 * 10^{-56}$	$2.20 * 10^{-56}$

For the small value of s ($s = 11$), the error converges slowly to zero as K increases: K must be chosen at least equal to 160 to have good approximations of the upper bound and the lower bounds. The probability $\bar{F}_{10}(11)$ is calculated with an error smaller than 10^{-15} when $K = 200$. For larger values of s, the upper bounds and the lower bounds of $\bar{F}_{10}(s)$ ($s = 50$ and $s = 1000$) may be calculated by using smaller values of K because the errors decrease very fast as s increases.

2.6 Intermediary proofs

Proof of Proposition 2.2

For $\alpha_1, \alpha_2 \in \mathbb{R}_+ \backslash \mathbb{N}$, let

$$\delta(s, \alpha_1, \alpha_2) = s^{\alpha_1+\alpha_2} \eta_{0,\infty}(s, \alpha_2, \alpha_1). \tag{2.4}$$

We first want to prove that $\varsigma((\beta_1+\beta_2),(\alpha_1,\beta_1),(\alpha_2,\beta_2))$ does not depend on (β_1,β_2). We have

$$\varsigma((\beta_1+\beta_2),(\alpha_1,\beta_1),(\alpha_2,\beta_2)) = \frac{(\beta_1+\beta_2)^{\alpha_1+\alpha_2}}{\beta_1^{\alpha_1}\beta_2^{\alpha_2}} - \delta\left(\frac{\beta_1+\beta_2}{\beta_2},\alpha_1,\alpha_2\right) - \delta\left(\frac{\beta_1+\beta_2}{\beta_1},\alpha_2,\alpha_1\right).$$

Let

$$A(z) = z^{\alpha_1+\alpha_2}(z-1)^{-\alpha_2} - \delta(z,\alpha_1,\alpha_2) - \delta\left(\frac{z}{z-1},\alpha_2,\alpha_2\right)$$

such that

$$\varsigma((\beta_1+\beta_2),(\alpha_1,\beta_1),(\alpha_2,\beta_2)) = A\left(\frac{\beta_1+\beta_2}{\beta_2}\right).$$

Since

$$\begin{aligned}\frac{\partial}{\partial z}\delta(z,\alpha_1,\alpha_2) &= \alpha_1 z^{\alpha_1-1}\left(1+\frac{\alpha_2}{1!}z^{-1}+\frac{\alpha_2(\alpha_2+1)}{2!}z^{-2}+\ldots\right)\\ &= \alpha_1 z^{\alpha_1-1}(1-z^{-1})^{-\alpha_2} = \alpha_1 z^{\alpha_1+\alpha_2-1}(z-1)^{-\alpha_2},\end{aligned}$$

we have

$$\begin{aligned}A'(z) &= (\alpha_1+\alpha_2)z^{\alpha_1+\alpha_2-1}(z-1)^{-\alpha_2} - \alpha_2(z-1)^{-(\alpha_2+1)}z^{\alpha_1+\alpha_2} - \alpha_1 z^{\alpha_1+\alpha_2-1}(z-1)^{-\alpha_2}\\ &\quad +\alpha_2 z^{\alpha_1+\alpha_2-1}(z-1)^{-(\alpha_2+1)}\\ &= 0\end{aligned}$$

for every $z>1$. It means that $A(z)$ is constant for every $z>1$. Hence we have

$$A\left(\frac{\beta_1+\beta_2}{\beta_2}\right) = A(2) = 2^{\alpha_1+\alpha_2} - \delta(2,\alpha_1,\alpha_2) - \delta(2,\alpha_1,\alpha_2).$$

and the result follows.

Note that for $\alpha_1,\alpha_2 \in \mathbb{R}_+\backslash\mathbb{N}$

$$\begin{aligned}\delta(2,\alpha_1,\alpha_2) &= 2^{\alpha_1} + 2^{(\alpha_1-1)}\alpha_2\frac{\alpha_1}{\alpha_1-1} + 2^{(\alpha_1-2)}\frac{\alpha_2(\alpha_2+1)}{2!}\frac{\alpha_1}{\alpha_1-2}+\ldots\\ &= 2^{\alpha_1+\alpha_2} + \frac{\alpha_2}{\alpha_1-1}\delta(2,\alpha_1-1,\alpha_2+1),\end{aligned}$$

and then

$$\begin{aligned}\delta(2,\alpha_1,\alpha_2+1) &= 2^{\alpha_1} + 2^{(\alpha_1-1)}(\alpha_2+1)\frac{\alpha_1}{\alpha_1-1} + 2^{(\alpha_1-2)}\frac{(\alpha_2+1)(\alpha_2+2)}{2!}\frac{\alpha_1}{\alpha_1-2}+\ldots\\ &= \delta(2,\alpha_1,\alpha_2) + \frac{\alpha_1}{\alpha_1-1}\,\delta(2,\alpha_1-1,\alpha_2+1)\\ &= \frac{\alpha_1+\alpha_2}{\alpha_2}\,\delta(2,\alpha_1,\alpha_2) - \frac{\alpha_1}{\alpha_2}2^{\alpha_1+\alpha_2}. \qquad (2.5)\end{aligned}$$

We now want to prove that

$$\begin{aligned} c_2(\alpha_1,\alpha_2) &= \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}{\Gamma(1-\alpha_1-\alpha_2)} = (\alpha_1+\alpha_2-1)\,B\,(1-\alpha_1,1-\alpha_2) \\ &= 2^{\alpha_1+\alpha_2} - \delta(2,\alpha_1,\alpha_2) - \delta(2,\alpha_2,\alpha_1). \end{aligned}$$

Let us begin with the case $0<\alpha_1<1$ and $0<\alpha_2<1$. We have

$$\begin{aligned} B\,(1-\alpha_1,1-\alpha_2) &= \int_0^1 x^{-\alpha_1}(1-x)^{-\alpha_2}dx = \int_0^{1/2} x^{-\alpha_1}(1-x)^{-\alpha_2}dx + \int_0^{1/2} x^{-\alpha_2}(1-x)^{-\alpha_1}dx \\ &= \int_0^{1/2} x^{-\alpha_1}\sum_{k=0}^{\infty} h(\alpha_2,k)x^k dx + \int_0^{1/2} x^{-\alpha_2}\sum_{k=0}^{\infty} h(\alpha_1,k)x^k dx \\ &= -\left(\sum_{k=0}^{\infty} h(\alpha_2,k)\frac{2^{\alpha_1-1-k}}{\alpha_1-1-k} + \sum_{k=0}^{\infty} h(\alpha_1,k)\frac{2^{\alpha_2-1-k}}{\alpha_2-1-k}\right) \\ &= -\left(\frac{\delta(2,\alpha_1-1,\alpha_2)}{\alpha_1-1} + \frac{\delta(2,\alpha_2-1,\alpha_1)}{\alpha_2-1}\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} c_2(\alpha_1,\alpha_2) &= (\alpha_1+\alpha_2-1)\,B\,(1-\alpha_1,1-\alpha_2) \\ &= -(\alpha_1+\alpha_2-1)\left(\frac{\delta(2,\alpha_1-1,\alpha_2)}{\alpha_1-1} + \frac{\delta(2,\alpha_2-1,\alpha_1)}{\alpha_2-1}\right) \\ &= -(\alpha_1+\alpha_2-1)\left(\frac{\delta(2,\alpha_1,\alpha_2)-2^{\alpha_1+\alpha_2-1}}{\alpha_1+\alpha_2-1} + \frac{\delta(2,\alpha_2,\alpha_1)-2^{\alpha_1+\alpha_2-1}}{\alpha_1+\alpha_2-1}\right) \\ &= 2^{\alpha_1+\alpha_2} - \delta(2,\alpha_1,\alpha_2) - \delta(2,\alpha_2,\alpha_1). \end{aligned}$$

For $\alpha_1,\alpha_2\in\mathbb{R}_+\backslash\mathbb{N}$, let us now define

$$d(\alpha_1,\alpha_2) = 2^{\alpha_1+\alpha_2} - \delta(2,\alpha_1,\alpha_2) - \delta(2,\alpha_2,\alpha_1) = \varsigma(2,(\alpha_1,1),(\alpha_2,1))$$

We deduce by (2.5) that

$$\begin{aligned} d(\alpha_1+1,\alpha_2) &= 2^{\alpha_1+\alpha_2+1} - \delta(2,\alpha_1+1,\alpha_2) - \delta(2,\alpha_2,\alpha_1+1) \\ &= -\frac{\alpha_2}{\alpha_1}\delta(2,\alpha_1,\alpha_2+1) - \delta(2,\alpha_2,\alpha_1+1) \\ &= -\frac{\alpha_2}{\alpha_1}\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\delta(2,\alpha_1,\alpha_2) - \frac{\alpha_1}{\alpha_2}2^{\alpha_1+\alpha_2}\right) - \frac{\alpha_1+\alpha_2}{\alpha_1}\left(\delta(2,\alpha_2,\alpha_1) + \frac{\alpha_2}{\alpha_1}2^{\alpha_1+\alpha_2}\right) \\ &= \frac{\alpha_1+\alpha_2}{\alpha_1}\left(2^{\alpha_1+\alpha_2} - \delta(2,\alpha_1,\alpha_2) - \delta(2,\alpha_2,\alpha_1)\right) \\ &= \frac{\alpha_1+\alpha_2}{\alpha_1}d(\alpha_1,\alpha_2). \end{aligned}$$

Since this property is also shared by

$$c_2(\alpha_1, \alpha_2) = \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}{\Gamma(1-\alpha_1-\alpha_2)},$$

we conclude that $c_2(\alpha_1, \alpha_2) = d(\alpha_1, \alpha_2)$ for $\alpha_1, \alpha_2 \in \mathbb{R}_+ \backslash \mathbb{N}$.

Proof of Proposition 2.3

We have, for $s \geq 2$,

$$\begin{aligned} P\left(X_1 + X_2 > s\right) &= P(X_1 > s-1) + P(X_1 < s-1, X_1 + X_2 > s) \\ &= P(X_1 > s-1) + \int_1^{s-1} \alpha_1 x^{-(\alpha_1+1)} (s-x)^{-\alpha_2} dx \\ &= P(X_1 > s-1) + \alpha_1 s^{-(\alpha_1+\alpha_2)} \int_{s^{-1}}^{1-s^{-1}} u^{-(\alpha_1+1)} (1-u)^{-\alpha_2} du. \quad (2.6) \end{aligned}$$

Let

$$g(s) = \int_{s^{-1}}^{1-s^{-1}} u^{-(\alpha_1+1)} (1-u)^{-\alpha_2} du, \qquad s \geq 2.$$

Such an integral does not have an explicit expression in general, but it can be written as a series expansion of power of s^{-1}. The function g is differentiable for every $s > 2$ and

$$\begin{aligned} g'(s) &= \frac{1}{s^2}\left((1-1/s)^{-(\alpha_1+1)}\ (1/s)^{-\alpha_2}\right) + \frac{1}{s^2}\left((1/s)^{-(\alpha_1+1)}(1-1/s)^{-\alpha_2}\right) \\ &= s^{\alpha_1+\alpha_2-1}\left((s-1)^{-(\alpha_1+1)} + (s-1)^{-\alpha_2}\right). \end{aligned}$$

Since $(s-1)^{-\theta}$ can be developed as a series expansion of s^{-1} for $\theta > 0$ in the following way

$$(s-1)^{-\theta} = \sum_{k=0}^{\infty} \frac{1}{kB(\theta, k)} s^{-(\theta+k)}, \qquad s > 1,$$

we deduce that

$$\begin{aligned} g'(s) &= s^{(\alpha_1+\alpha_2-1)}\left(\sum_{k=0}^{\infty} \frac{1}{kB(\alpha_1+1, k)} s^{-(\alpha_1+1+k)} + \sum_{k=0}^{\infty} \frac{1}{kB(\alpha_2, k)} s^{-(\alpha_2+k)}\right) \\ &= \left(s^{(\alpha_1-1)} + \alpha_2 s^{(\alpha_1-2)} + \frac{\alpha_2(\alpha_2+1)}{2!} s^{(\alpha_1-3)} + \ldots\right) + \\ &\quad + \left(s^{(\alpha_2-2)} + (\alpha_1+1) s^{(\alpha_2-3)} + \frac{(\alpha_1+1)(\alpha_1+2)}{2!} s^{(\alpha_2-4)} + \ldots\right). \end{aligned}$$

With the assumption that α_1 and α_2 are not integers, we derive that

$$\begin{aligned} g(s) &= \left(\frac{s^{\alpha_1}}{\alpha_1} + \alpha_2 \frac{s^{\alpha_1-1}}{\alpha_1 - 1} + \frac{\alpha_2(\alpha_2+1)}{2!}\frac{s^{\alpha_1-2}}{\alpha_1-2} + \dots\right) \\ &\quad + \left(\frac{s^{\alpha_2-1}}{\alpha_2-1} + (\alpha_1+1)\frac{s^{\alpha_2-2}}{\alpha_2-2} + \frac{(\alpha_1+1)(\alpha_1+2)}{2!}\frac{s^{\alpha_2-3}}{\alpha_2-3} + \dots\right) + \gamma_0 \end{aligned}$$

where γ_0 is a constant which only depends on α_1 and α_2 and is determined by the condition that $g(2) = 0$.

Replacing g in Equation (2.6), we have

$$\begin{aligned} P(X_1 + X_2 > s) &= (s-1)^{-\alpha_1} + \alpha_1 s^{-(\alpha_1+\alpha_2)} g(s) \\ &= \left(s^{-\alpha_1} + \frac{\alpha_2}{\alpha_2-1}\frac{\alpha_1}{1!}s^{-(\alpha_1+1)} + \frac{\alpha_2}{\alpha_2-2}\frac{\alpha_1(\alpha_1+1)}{2!}s^{-(\alpha_1+2)} + \dots\right) \\ &\quad + \left(s^{-\alpha_2} + \frac{\alpha_1}{\alpha_1-1}\frac{\alpha_2}{1!}s^{-(\alpha_2+1)}\frac{\alpha_1}{\alpha_1-2}\frac{\alpha_2(\alpha_2+1)}{2!}s^{-(\alpha_2+2)} + \dots\right) \\ &\quad + c_2(\alpha_1,\alpha_2)s^{-(\alpha_1+\alpha_2)} \\ &= \eta_{0,\infty}(s,\alpha_1,\alpha_2) + \eta_{0,\infty}(s,\alpha_2,\alpha_1) + c_2(\alpha_1,\alpha_2)s^{-(\alpha_1+\alpha_2)} \end{aligned}$$

where $c_2(\alpha_1,\alpha_2) = \alpha_1\gamma_0$ does not depend on s and is such that

$$\eta_{0,\infty}(2,\alpha_1,\alpha_2) + \eta_{0,\infty}(2,\alpha_2,\alpha_1) + c_2(\alpha_1,\alpha_2)2^{-(\alpha_1+\alpha_2)} = 1,$$

since $P(X_1 + X_2 > 2) = 1$.

Proof of Proposition 2.4

We can write that

$$\begin{aligned} P(X_1 + X_2 > s) &= P(X_1 > s - \beta_2) + P(X_1 < s - \beta_2, X_1 + X_2 > s) \\ &= \beta_1^{\alpha_1}(s-\beta_2)^{-\alpha_1} + \int_{\beta_1}^{s-\beta_2} \alpha_1\beta_1^{\alpha_1}x^{-(\alpha_1+1)}\beta_2^{\alpha_2}(s-x)^{-\alpha_2}dx \qquad (2.7) \\ &= P(X_1 > s-\beta_2) + \beta_1^{\alpha_1}\beta_2^{\alpha_2}\alpha_1 s^{-(\alpha_1+\alpha_2)}\int_{\beta_1}^{s-\beta_2} u^{-(\alpha_1+1)}(1-u)^{-\alpha_2}du. \end{aligned}$$

Similarly to the proof of Proposition 2.3, we derive the series expansion of the probability survival function

$$\begin{aligned} P\left(X_1+X_2>s\right) &= \beta_1^{\alpha_1}\left(s^{-\alpha_1}+\alpha_1\frac{\alpha_2}{\alpha_2-1}\beta_2 s^{-(\alpha_1+1)}+\ldots\right)+ \\ &\quad +\beta_2^{\alpha_2}\left(s^{-\alpha_2}+\alpha_2\frac{\alpha_1}{\alpha_1-1}\beta_1 s^{-(\alpha_2+1)}+\ldots\right)+ \\ &\quad +\beta_1^{\alpha_1}\beta_2^{\alpha_2}\gamma_1 s^{-(\alpha_1+\alpha_2)} \\ &= \beta_1^{\alpha_1}\beta_2^{\alpha_2}s^{-(\alpha_1+\alpha_2)}\left(\delta\left(\frac{s}{\beta_2},\alpha_2,\alpha_1\right)+\delta\left(\frac{s}{\beta_1},\alpha_1,\alpha_2\right)+\gamma_1\right) \end{aligned}$$

where γ_1 is a constant which does not depend on s, and δ is the function defined in Equation (2.4). Since $P(X_1+X_2>\beta_1+\beta_2)=1$, we have

$$\gamma_1=\frac{(\beta_1+\beta_2)^{(\alpha_1+\alpha_2)}}{\beta_1^{\alpha_1}\beta_2^{\alpha_2}}-\delta\left(\frac{\beta_1+\beta_2}{\beta_2},\alpha_2,\alpha_1\right)-\delta\left(\frac{\beta_1+\beta_2}{\beta_2},\alpha_1,\alpha_2\right).$$

By Proposition 2.2, we deduce that $\gamma_1=c_2(\alpha_1,\alpha_2)$.

Proof of Proposition 2.5

The survival probability distribution function of the sum of H_1 and H_2 may be written as

$$\begin{aligned} P\left(H_1+H_2>s\right) &= P(H_1>s-r_2)+P(H_1<(s-r_2),H_1+H_2>s) \\ &= \sum_{i=1}^{\infty}\gamma_{1,i}(s-r_2)^{-\alpha_{1,i}}+\int_{r_1}^{s-r_2}\sum_{i=1}^{\infty}\gamma_{1,i}\alpha_{1,i}x^{-(\alpha_{1,i}+1)}\sum_{i=1}^{\infty}\gamma_{2,i}(s-x)^{-\alpha_{2,i}}dx \\ &= \sum_{i=1}^{\infty}\gamma_{1,i}(s-r_2)^{-\alpha_{1,i}}+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\gamma_{1,i}\gamma_{2,j}\int_{r_1}^{s-r_2}\alpha_{1,i}x^{-(\alpha_{1,i}+1)}(s-x)^{-\alpha_{2,i}}dx. \end{aligned}$$

Since $\sum_{j=1}^{\infty}\gamma_{2,j}r_2^{-\alpha_{2,j}}=1$, we have by Equation (2.7)

$$\begin{aligned} P\left(H_1+H_2>s\right) &= \sum_{i=1}^{\infty}\gamma_{1,i}(s-r_2)^{-\alpha_{1,i}}\sum_{j=1}^{\infty}\frac{\gamma_{2,j}}{r_2^{\alpha_{2,j}}}+\sum_{i,j=1}^{\infty}\gamma_{1,i}\gamma_{2,j}\int_{r_1}^{s-r_2}\alpha_{1,i}x^{-(\alpha_{1,i}+1)}(s-x)^{-\alpha_{2,i}}dx \\ &= \sum_{i,j=1}^{\infty}\frac{\gamma_{1,i}}{r_1^{\alpha_{1,i}}}\frac{\gamma_{2,j}}{r_2^{\alpha_{2,j}}}\left(r_1^{\alpha_{1,i}}(s-r_2)^{-\alpha_{1,i}}+r_1^{\alpha_{1,i}}r_2^{\alpha_{2,j}}\int_{r_1}^{s-r_2}\alpha_{1,i}x^{-(\alpha_{1,i}+1)}(s-x)^{-\alpha_{2,i}}dx\right) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\gamma_{1,i}}{r_1^{\alpha_{1,i}}}\frac{\gamma_{2,j}}{r_2^{\alpha_{2,j}}}P\left(X_i+Y_j>s\right). \end{aligned}$$

Proof of Proposition 2.6

From Proposition 2.4, we have

$$\begin{aligned} P\left(X_1+X_2>s\right) &= \sum_{j=0}^{\infty}\lambda_{1,j}\left(\beta_1+\beta_2\right)^{\alpha_1+j}s^{-(\alpha_1+j)}+\sum_{j=0}^{\infty}\lambda_{2,j}\left(\beta_1+\beta_2\right)^{\alpha_2+j}s^{-(\alpha_2+j)} \\ &\quad +\lambda_3\left(\beta_1+\beta_2\right)^{\alpha_1+\alpha_2}s^{-(\alpha_1+\alpha_2)} \end{aligned}$$

where

$$\begin{aligned} \lambda_{1,j} &= \frac{\beta_1^{\alpha_1}h(\alpha_1,j)w_1((\alpha_2,\beta_2),j)}{(\beta_1+\beta_2)^{\alpha_1+j}}, \qquad j\geq 1, \\ \lambda_{2,j} &= \frac{\beta_2^{\alpha_2}h(\alpha_2,j)w_1((\alpha_1,\beta_1),j)}{(\beta_1+\beta_2)^{\alpha_2+j}}, \qquad j\geq 1, \\ \lambda_3 &= \frac{\beta_1^{\alpha_1}\beta_2^{\alpha_2}c_2(\alpha_1,\alpha_2)}{(\beta_1+\beta_2)^{\alpha_1+\alpha_2}}. \end{aligned}$$

Let $Y_{1,j}(\alpha_1+j,\beta_1+\beta_2)$, $Y_{2,j}(\alpha_1+j,\beta_1+\beta_2)$, $Y_3(\alpha_1+\alpha_2,\beta_1+\beta_2)$ be independent Pareto random variables, independent of X_3. We have

$$P\left(X_1+X_2>s\right)=\sum_{j=0}^{\infty}\lambda_{1,j}P\left(Y_{1,j}>s\right)+\sum_{j=0}^{\infty}\lambda_{2,j}P\left(Y_{2,j}>s\right)+\lambda_3P\left(Y_3>s\right).$$

Using again Proposition 2.4, we deduce that

$$\begin{aligned} P\left(Y_{1,j}+X_3>s\right) &= \left(\beta_1+\beta_2\right)^{\alpha_1+j}\sum_{k=0}^{\infty}h(\alpha_1+j,k)w_1((\alpha_3,\beta_3),k)s^{-(\alpha_1+j+k)} \\ &\quad +\beta_3^{\alpha_3}\sum_{k=0}^{\infty}h(\alpha_3,k)w_1((\alpha_1+j,\beta_1+\beta_2),k)s^{-(\alpha_3+k)} \\ &\quad +\left(\beta_1+\beta_2\right)^{\alpha_1+j}\beta_3^{\alpha_3}c_2(\alpha_1+j,\alpha_3)s^{-(\alpha_1+j+\alpha_3)} \end{aligned}$$

and

$$\begin{aligned} P\left(Y_{2,j}+X_3>s\right) &= \left(\beta_1+\beta_2\right)^{\alpha_2+j}\sum_{k=0}^{\infty}h(\alpha_2+j,k)w_1((\alpha_3,\beta_3),k)s^{-(\alpha_2+j+k)} \\ &\quad +\beta_3^{\alpha_3}\sum_{k=0}^{\infty}h(\alpha_3,k)w_1((\alpha_2+j,\beta_1+\beta_2),k)s^{-(\alpha_3+k)} \\ &\quad +\left(\beta_1+\beta_2\right)^{\alpha_2+j}\beta_3^{\alpha_3}c_2(\alpha_2+j,\alpha_3)s^{-(\alpha_2+j+\alpha_3)} \end{aligned}$$

and

$$\begin{aligned} P\left(Y_3+X_3>s\right) &= \left(\beta_1+\beta_2\right)^{\alpha_1+\alpha_2}\sum_{k=0}^{\infty}h(\alpha_1+\alpha_2,k)w_1((\alpha_3,\beta_3),k)s^{-(\alpha_1+\alpha_2+k)} \\ &\quad +\beta_3^{\alpha_3}\sum_{k=0}^{\infty}h(\alpha_3,k)w_1((\alpha_1+\alpha_2,\beta_1+\beta_2),k)s^{-(\alpha_3+k)} \\ &\quad +\left(\beta_1+\beta_2\right)^{\alpha_1+\alpha_2}\beta_3^{\alpha_3}c_2(\alpha_1+\alpha_2,\alpha_3)s^{-(\alpha_1+\alpha_2+\alpha_3)}. \end{aligned}$$

By Proposition 2.5, we have that

$$\begin{aligned} & P\left(X_1+X_2+X_3>s\right) \\ = & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_{1,j}\left(\beta_1+\beta_2\right)^{\alpha_1+j}h(\alpha_1+j,k)w_1((\alpha_3,\beta_3),k)s^{-(\alpha_1+j+k)} \\ & +\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_{2,j}\left(\beta_1+\beta_2\right)^{\alpha_2+j}h(\alpha_2+j,k)w_1((\alpha_3,\beta_3),k)s^{-(\alpha_2+j+k)} \\ & +\beta_3^{\alpha_3}\sum_{k=0}^{\infty}h(\alpha_3,k)\Big(\lambda_3 w_1((\alpha_1+\alpha_2,\beta_1+\beta_2),k) \\ & +\sum_{j=0}^{\infty}\lambda_{1,j}w_1((\alpha_1+j,\beta_1+\beta_2),k)+\sum_{j=0}^{\infty}\lambda_{2,j}w_1((\alpha_2+j,\beta_1+\beta_2),k)\Big)s^{-(\alpha_3+k)} \\ & +\lambda_3\left(\beta_1+\beta_2\right)^{\alpha_1+\alpha_2}\sum_{k=0}^{\infty}h(\alpha_1+\alpha_2,k)w_1((\alpha_3,\beta_3),k)s^{-(\alpha_1+\alpha_2+k)} \\ & +\sum_{j=0}^{\infty}\lambda_{1,j}\left(\beta_1+\beta_2\right)^{\alpha_1+j}\beta_3^{\alpha_3}c_2(\alpha_1+j,\alpha_3)s^{-(\alpha_1+\alpha_3+j)} \\ & +\sum_{j=0}^{\infty}\lambda_{2,j}\left(\beta_1+\beta_2\right)^{\alpha_2+j}\beta_3^{\alpha_3}c_2(\alpha_2+j,\alpha_3)s^{-(\alpha_2+\alpha_3+j)} \\ & +\lambda_3\left(\beta_1+\beta_2\right)^{\alpha_1+\alpha_2}\beta_3^{\alpha_3}c_2(\alpha_1+\alpha_2,\alpha_3)s^{-(\alpha_1+\alpha_2+\alpha_3)}. \end{aligned}$$

Then note that

$$\begin{aligned} & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_{1,j}\left(\beta_1+\beta_2\right)^{\alpha_1+j}c_1(\alpha_1+j)h(\alpha_1+j,k)w_1((\alpha_3,\beta_3),k)s^{-(\alpha_1+j+k)} \\ = & \sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}\lambda_{1,j}\left(\beta_1+\beta_2\right)^{\alpha_1+j}c_1(\alpha_1+j)h(\alpha_1+j,l-j)w_1((\alpha_3,\beta_3),l-j)\right)s^{-(\alpha_1+l)} \end{aligned}$$

and that

$$\begin{aligned}
&\sum_{j=0}^{l} \lambda_{1,j} \left(\beta_1+\beta_2\right)^{\alpha_1+j} c(\alpha_1+j) h(\alpha_1+j, l-j) w_1((\alpha_3,\beta_3), l-j) \\
= &\sum_{j=0}^{l} \frac{c(\alpha_1) h(\alpha_1, j) w_1((\alpha_2,\beta_2), j)}{\left(\beta_1+\beta_2\right)^{\alpha_1+j}} \left(\beta_1+\beta_2\right)^{\alpha_1+j} \frac{(\alpha_1+j)(\alpha_1+j+1)\ldots(\alpha_1+l-1)}{(l-j)!} \frac{\alpha_3 \beta_3^{l-j}}{\alpha_3-l+j} \\
= &\beta_1^{\alpha_1} \sum_{j=0}^{l} \frac{\alpha_1(\alpha_1+1)\ldots(\alpha_1+j-1)}{j!} \frac{\alpha_2 \beta_2^{j}}{\alpha_2-j} \frac{(\alpha_1+j)(\alpha_1+j+1)\ldots(\alpha_1+l-1)}{(l-j)!} \frac{\alpha_3 \beta_3^{l-j}}{\alpha_3-l+j} \\
= &\beta_1^{\alpha_1} h(\alpha_1, l) \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} \frac{\alpha_2 \beta_2^{j}}{\alpha_2-j} \frac{\alpha_3 \beta_3^{l-j}}{\alpha_3-l+j}.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\lambda_3 \left(\beta_1+\beta_2\right)^{\alpha_1+\alpha_2} \beta_3^{\alpha_3} c_2(\alpha_1+\alpha_2, \alpha_3) \\
= &\frac{\beta_1^{\alpha_1} \beta_2^{\alpha_2} c(\alpha_1, \alpha_2)}{\left(\beta_1+\beta_2\right)^{\alpha_1+\alpha_2}} \left(\beta_1+\beta_2\right)^{\alpha_1+\alpha_2} \beta_3^{\alpha_3} c_2(\alpha_1+\alpha_2, \alpha_3) \\
= &\beta_1^{\alpha_1} \beta_2^{\alpha_2} \beta_3^{\alpha_3} c_2(\alpha_1, \alpha_2) c_2(\alpha_1+\alpha_2, \alpha_3)
\end{aligned}$$

and

$$\begin{aligned}
\lambda_3 \left(\beta_1+\beta_2\right)^{\alpha_1+\alpha_2} c_1(\alpha_1+\alpha_2) &= \frac{\beta_1^{\alpha_1} \beta_2^{\alpha_2} c(\alpha_1, \alpha_2)}{\left(\beta_1+\beta_2\right)^{\alpha_1+\alpha_2}} \left(\beta_1+\beta_2\right)^{\alpha_1+\alpha_2} c_1(\alpha_1+\alpha_2) \\
&= \beta_1^{\alpha_1} \beta_2^{\alpha_2} c_2(\alpha_1, \alpha_2) c(\alpha_1+\alpha_2) \\
&= \beta_1^{\alpha_1} \beta_2^{\alpha_2} c_2(\alpha_1, \alpha_2).
\end{aligned}$$

By symmetry with respect to (α_1,β_1), (α_2,β_2) and (α_3,β_3), we deduce that

$$\begin{aligned}
P\left(X_1+X_2+X_3>s\right) &= \sum_{l=0}^{\infty}\beta_1^{\alpha_1}h(\alpha_1,l)\left(\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\frac{\alpha_2\beta_2^j}{\alpha_2-j}\frac{\alpha_3\beta_3^{l-j}}{\alpha_3-l+j}\right)s^{-(\alpha_1+l)}\\
&+\sum_{l=0}^{\infty}\beta_2^{\alpha_2}h(\alpha_2,l)\left(\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\frac{\alpha_1\beta_1^j}{\alpha_1-j}\frac{\alpha_3\beta_3^{l-j}}{\alpha_3-l+j}\right)s^{-(\alpha_2+l)}\\
&+\sum_{l=0}^{\infty}\beta_3^{\alpha_3}h(\alpha_3,l)\left(\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\frac{\alpha_1\beta_1^j}{\alpha_1-j}\frac{\alpha_2\beta_2^{l-j}}{\alpha_2-l+j}\right)s^{-(\alpha_3+l)}\\
&+\beta_1^{\alpha_1}\beta_2^{\alpha_2}c_2(\alpha_1,\alpha_2)\sum_{k=0}^{\infty}h(\alpha_1+\alpha_2,k)w_1((\alpha_3,\beta_3),k)s^{-(\alpha_1+\alpha_2+k)}\\
&+\beta_1^{\alpha_1}\beta_3^{\alpha_3}c_2(\alpha_1,\alpha_3)\sum_{k=0}^{\infty}h(\alpha_1+\alpha_3,k)w_1((\alpha_2,\beta_2),k)s^{-(\alpha_1+\alpha_3+k)}\\
&+\beta_2^{\alpha_2}\beta_3^{\alpha_3}c_2(\alpha_2,\alpha_3)\sum_{k=0}^{\infty}h(\alpha_2+\alpha_3,k)w_1((\alpha_1,\beta_1),k)s^{-(\alpha_2+\alpha_3+k)}\\
&+\beta_1^{\alpha_1}\beta_2^{\alpha_2}\beta_3^{\alpha_3}c_2(\alpha_1,\alpha_2)c_2(\alpha_1+\alpha_2,\alpha_3)s^{-(\alpha_1+\alpha_2+\alpha_3)}.
\end{aligned}$$

It suffices to note that $c_3(\alpha_1,\alpha_2,\alpha_3)=c_2(\alpha_1,\alpha_2)c_2(\alpha_1+\alpha_2,\alpha_3)$ and

$$w_2((\alpha_2,\beta_2),(\alpha_3,\beta_3),l)=\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\frac{\alpha_2\beta_2^j}{\alpha_2-j}\frac{\alpha_3\beta_3^{l-j}}{\alpha_3-l+j},$$

to have

$$\begin{aligned}
P\left(X_1+X_2+X_3>s\right) &= \beta_1^{\alpha_1}\sum_{k=0}^{\infty}h(\alpha_1,k)w_2((\alpha_2,\beta_2),(\alpha_3,\beta_3),k)s^{-(\alpha_1+k)}\\
&+\beta_2^{\alpha_2}\sum_{k=0}^{\infty}h(\alpha_2,k)w_2((\alpha_1,\beta_1),(\alpha_3,\beta_3),k)s^{-(\alpha_2+k)}\\
&+\beta_3^{\alpha_3}\sum_{k=0}^{\infty}h(\alpha_3,k)w_2((\alpha_1,\beta_1),(\alpha_2,\beta_2),k)s^{-(\alpha_3+k)}\\
&+\beta_1^{\alpha_1}\beta_2^{\alpha_2}c_2(\alpha_1,\alpha_2)\sum_{k=0}^{\infty}h(\alpha_1+\alpha_2,k)w_1((\alpha_3,\beta_3),k)s^{-(\alpha_1+\alpha_2+k)}\\
&+\beta_1^{\alpha_1}\beta_3^{\alpha_3}c_2(\alpha_1,\alpha_3)\sum_{k=0}^{\infty}h(\alpha_1+\alpha_3,k)w_1((\alpha_2,\beta_2),k)s^{-(\alpha_1+\alpha_3+k)}\\
&+\beta_2^{\alpha_2}\beta_3^{\alpha_3}c_2(\alpha_2,\alpha_3)\sum_{k=0}^{\infty}h(\alpha_2+\alpha_3,k)w_1((\alpha_1,\beta_1),k)s^{-(\alpha_2+\alpha_3+k)}\\
&+\beta_1^{\alpha_1}\beta_2^{\alpha_2}\beta_3^{\alpha_3}c_3(\alpha_1,\alpha_2,\alpha_3)s^{-(\alpha_1+\alpha_2+\alpha_3)}.
\end{aligned}$$

Proof of Proposition 2.7

Let $X_1(\alpha_1, 1)$, $X_2(\alpha_2, 1)$ be two independent Pareto random variables such that $\alpha_1 \in (m_1 - 1, m_1 + 1) \setminus \{m_1\}, \alpha_2 \in (m_2 - 1, m_2 + 1) \setminus \{m_2\}$. By Proposition 2.3, we have, for $s \geq 2$,

$$
\begin{aligned}
& P(X_1 + X_2 > s) \\
= & \sum_{k=0}^{m_2-1} \frac{\alpha_2}{(\alpha_2 - k)} \frac{1}{kB(\alpha_1, k)} s^{-(\alpha_1+k)} + \sum_{k=0}^{m_1-1} \frac{\alpha_1}{(\alpha_1 - k)} \frac{1}{kB(\alpha_2, k)} s^{-(\alpha_2+k)} \\
+ & \frac{\alpha_2}{(\alpha_2 - m_2)} \frac{1}{m_2 B(\alpha_1, m_2)} \left(s^{-(\alpha_1+m_2)} - 2^{-(\alpha_1+m_2)} \left(\frac{s}{2}\right)^{-(\alpha_1+\alpha_2)} \right) \\
+ & \frac{\alpha_1}{(\alpha_1 - k)} \frac{1}{kB(\alpha_2, k)} \left(s^{-(\alpha_2+k)} - 2^{-(\alpha_2+k)} \left(\frac{s}{2}\right)^{-(\alpha_1+\alpha_2)} \right) \\
+ & \sum_{k=m_2+1}^{\infty} \frac{\alpha_2}{(\alpha_2 - k)} \frac{1}{kB(\alpha_1, k)} s^{-(\alpha_1+k)} + \sum_{k=m_1+1}^{\infty} \frac{\alpha_1}{(\alpha_1 - k)} \frac{1}{kB(\alpha_2, k)} s^{-(\alpha_2+k)} \\
+ & \left(1 - \sum_{k\geq 0}^{k\neq m_2} \frac{\alpha_2}{(\alpha_2 - k)} \frac{1}{kB(\alpha_1, k)} 2^{-(\alpha_1+k)} - \sum_{k\geq 0}^{k\neq m_1} \frac{\alpha_1}{(\alpha_1 - k)} \frac{1}{kB(\alpha_2, k)} 2^{-(\alpha_2+k)} \right) \left(\frac{s}{2}\right)^{-(\alpha_1+\alpha_2)}.
\end{aligned}
$$

We let α_1 tend to m_1 and α_2 tend to m_2. We first have

$$
\sum_{k=0}^{m_2-1} \frac{\alpha_2}{(\alpha_2 - k)} \frac{1}{kB(\alpha_1, k)} s^{-(\alpha_1+k)} \to \sum_{k=0}^{m_2-1} \frac{m_2}{(m_2 - k)} \frac{1}{kB(m_1, k)} s^{-(m_1+k)} = \eta_{0,m_2-1}(s, m_1, m_2)
$$

and

$$
\frac{\alpha_2}{(\alpha_2 - m_2)} \frac{1}{m_2 B(\alpha_1, m_2)} \left(s^{-(\alpha_1+m_2)} - 2^{-(\alpha_1+m_2)} \left(\frac{s}{2}\right)^{-(\alpha_1+\alpha_2)} \right) \to \frac{1}{B(m_1, m_2)} s^{-(m_1+m_2)} \ln\left(\frac{s}{2}\right)
$$

Then

$$\begin{aligned}
& \sum_{k=m_2+1}^{\infty} \frac{\alpha_2}{(\alpha_2-k)} \frac{1}{kB(\alpha_1,k)} s^{-(\alpha_1+k)} \\
\to \quad & \sum_{k=m_2+1}^{\infty} \frac{m_2}{(m_2-k)} \frac{1}{kB(m_1,k)} s^{-(m_1+k)} \\
= \quad & -\frac{m_2}{(m_1-1)!} s^{-(m_1+m_2)} \sum_{k=m_2+1}^{\infty} \frac{1}{(k-m_2)} \frac{(m_1+k-1)!}{k!} s^{-(k-m_2)} \\
= \quad & -\frac{m_2}{(m_1-1)!} s^{-(m_1+m_2)} \sum_{k=m_2+1}^{\infty} \frac{1}{(k-m_2)} (k+1)...(m_1+k-1) s^{-(k-m_2)} \\
= \quad & -\frac{m_2}{(m_1-1)!} s^{-(m_1+m_2)} \sum_{k=m_2+1}^{\infty} \frac{1}{(k-m_2)} ((k-m_2)+m_2+1)...(k+m_1-1) s^{-(k-m_2)} \\
= \quad & -\frac{m_2}{(m_1-1)!} s^{-(m_1+m_2)} \sum_{j=0}^{m_1-1} \varphi_{j,m_1-1,m_2} \sum_{k=1}^{\infty} k^{j-1} s^{-k}.
\end{aligned}$$

The result follows.

Proof of Proposition 2.8

Let

$$f(s,m_1,m_2) = \int_{1/s}^{1-1/s} u^{-m_1}(1-u)^{-m_2} du, \qquad s \geq 2.$$

Since $1 = u + (1-u)$, we have

$$f(s,m_1,m_2) = f(s,m_1-1,m_2) + f(s,m_1,m_2-1).$$

Note that

$$\begin{aligned}
& f(s,m_1,m_2-1) \\
= \quad & \int_{1/s}^{1-1/s} u^{-m_1}(1-u)^{-(m_2-1)} du \\
= \quad & \left[\frac{u^{-(m_1-1)}}{-(m_1-1)} (1-u)^{-(m_2-1)} \right]_{1/s}^{1-1/s} + \frac{m_2-1}{m_1-1} \int_{1/s}^{1-1/s} u^{-(m_1-1)}(1-u)^{-m_2} du \\
= \quad & \frac{1}{m_1-1} s^{(m_1+m_2-2)} \left[(s-1)^{-(m_2-1)} - (s-1)^{-(m_1-1)} \right] + \frac{m_2-1}{m_1-1} f(s,m_1-1,m_2)
\end{aligned}$$

and thus

$$f(s,m_1,m_2) = \frac{1}{m_1-1} s^{(m_1+m_2-2)} \left[(s-1)^{-(m_2-1)} - (s-1)^{-(m_1-1)} \right] + \frac{m_1+m_2-2}{m_1-1} f(s,m_1-1,m_2).$$

On the one hand, we have from Equation (2.6)

$$P(X_1 + X_2 > s) = (s-1)^{-m_1} + m_1 s^{-(m_1+m_2)} f(s, m_1+1, m_2)$$

and therefore

$$P(X_1 + X_2 > s) = \frac{1}{s(s-1)^{m_1-1}} + \frac{1}{s(s-1)^{m_2-1}} + \frac{m_1+m_2-1}{s^{m_1+m_2}} f(s, m_1, m_2).$$

On the other hand, $f(s, m_1, m_2)$ can be calculated recursively as follows: $f(s, m_1, m_2)$ is written as a function of $f(s, m_1-1, m_2)$, and $f(s, m_1-1, m_2)$ as a function of $f(s, m_1-1, m_2-1)$:

$$\begin{aligned} f(s, m_1, m_2) &= \frac{m_1+m_2-2}{(m_1-1)(m_2-1)} s^{m_1+m_2-2} \\ &\times \left[\frac{1}{(s-1)^{m_1-1}} \left(\frac{m_1-1}{m_1+m_2-2} - \frac{1}{s} \right) + \frac{1}{(s-1)^{m_2-1}} \left(\frac{m_2-1}{m_1+m_2-2} - \frac{1}{s} \right) \right] \\ &+ \frac{(m_1+m_2-2)(m_1+m_2-3)}{(m_1-1)(m_2-1)} f(s, m_1-1, m_2-1). \end{aligned}$$

If $m_1 \geq m_2$, we finally derive that

$$\begin{aligned} P\left(X_1 + X_2 > s\right) &= \frac{1}{s(s-1)^{m_1-1}} + \frac{1}{s(s-1)^{m_2-1}} \\ &+ \sum_{i=1}^{m_2-1} \frac{(m_1+m_2-1)(m_1+m_2-2)\dots(m_1+m_2-2i)}{(m_1-1)(m_1-2)\dots(m_1-i)(m_2-1)(m_2-2)\dots(m_2-i)} s^{-2i} \\ &\times \left[\frac{1}{(s-1)^{m_1-i}} \left(\frac{m_1-i}{m_1+m_2-2i} - \frac{1}{s} \right) + \frac{1}{(s-1)^{m_2-i}} \left(\frac{m_2-i}{m_1+m_2-2i} - \frac{1}{s} \right) \right] \\ &+ \frac{(m_1+m_2-1)(m_1+m_2-2)\dots(m_1-m_2+1)}{(m_1-1)(m_1-2)\dots(m_1-m_2+1)(m_2-1)!} s^{-(m_1+m_2)} f(s, m_1-m_2+1, 1) \end{aligned}$$

where it is easely seen that, for $j \in \mathbb{N}$,

$$f(s, j+1, 1) = \sum_{k=1}^{j} \frac{s^k}{k} \left(1 - \frac{1}{(s-1)^k} \right) + 2\log\left(s-1\right).$$

Proof of Proposition 2.9

The proof follows the same lines as those of proofs of Proposition 2.7 and Proposition 2.8. In order to shorten the paper, the proof is not given here but it is, however, available upon request.

Proof of Proposition 2.11

For the proof of Proposition 2.11, we shall use the following proposition.

Proposition 2.12. *Let* $(a_k)_{k\geq 0}$ *and* $(b_k)_{k\geq 0}$ *two sequences of real numbers satisfying (i)* $\lim_{k\to\infty}|a_k| = \infty$ *and* $\lim_{k\to\infty}|b_k| = \infty$*, (ii)* $\lim_{k\to\infty} a_k/a_{k-1} = \alpha$ *and* $\lim_{k\to\infty} b_k/b_{k-1} = \beta$ *where* $\alpha > 0, \beta > 0$ *and* $\alpha + \beta > 1$*. If* $S_k = \sum_{i=0}^{k} C_k^i a_i b_{k-i}$*, then* $\lim_{k\to\infty} S_k/S_{k-1} = (\alpha+\beta)$.

Proof: Since $\alpha > 0$ and $\beta > 0$, $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ do not change sign for large k. We can assume that both sequences are positive from $k_0 > 0$ without loss of generality. Following Cauchy's criterion, the convergences $\lim_{k\to\infty} a_k/a_{k-1} = \alpha$ and $\lim_{k\to\infty} b_k/b_{k-1} = \beta$ are equivalent to $\lim_{k\to\infty}(a_k)^{1/k} = \alpha$ and $\lim_{k\to\infty}(b_k)^{1/k} = \beta$. Hence, for some positive $\varepsilon < (\alpha+\beta-1)/2$, there exits $k_1 > k_0$ such that, for $k \geq k_1$, we have: $(\alpha-\varepsilon)^k < a_k < (\alpha+\varepsilon)^k$ and $(\beta-\varepsilon)^k < b_k < (\beta+\varepsilon)^k$. For $k > 2k_1$ we can write

$$\begin{aligned}
S_k &= \sum_{i=0}^{k_1-1} C_k^i a_i b_{k-i} + \sum_{i=k-k_1+1}^{k} C_k^i a_i b_{k-i} + \sum_{i=k_1}^{k-k_1} C_k^i a_i b_{k-i} \\
&= \sum_{i=0}^{k_1-1} C_k^i a_i b_{k-i} + \sum_{i=0}^{k_1-1} C_k^i b_i a_{k-i} + \sum_{i=k_1}^{k-k_1} C_k^i a_i b_{k-i} \\
&\geq \sum_{i=0}^{k_1-1} C_k^i \left[(a_i - (\alpha-\varepsilon)^i) b_{k-i} + (b_i - (\beta-\varepsilon)^i) a_{k-i}\right] + \sum_{i=0}^{k} C_k^i (\alpha-\varepsilon)^i (\beta-\varepsilon)^{k-i} \\
&= \sum_{i=0}^{k_1-1} C_k^i \left[(a_i - (\alpha-\varepsilon)^i) b_{k-i} + (b_i - (\beta-\varepsilon)^i) a_{k-i}\right] + (\alpha+\beta-2\varepsilon)^k
\end{aligned}$$

Let $M = \max_{j=0,\ldots,k_1}\{|a_j - (\alpha-\varepsilon)^j|, |b_j - (\beta-\varepsilon)^j|\}$. We have

$$S_k \geq (\alpha+\beta-2\varepsilon)^k - M k^{k_1} \sum_{i=0}^{k_1-1} \left[(\alpha+\varepsilon)^{k-i} + (\beta+\varepsilon)^{k-i}\right].$$

On the other hand, since $(\alpha+\beta-2\varepsilon) > 1$, we have

$$\lim_{k\to\infty} \frac{1}{(\alpha+\beta-2\varepsilon)^k} \left(k^{k_1} \sum_{i=0}^{k_1-1} \left[(\alpha+\varepsilon)^{k-i} + (\beta+\varepsilon)^{k-i}\right] \right) = 0.$$

Therefore for any positive $\gamma < 1$, we have, for large k, $S_k \geq (1-\gamma)(\alpha+\beta-2\varepsilon)^k$.

Similarly, we can derive, for large k, that $S_k \leq (1+\gamma)(\alpha+\beta+2\varepsilon)^k$. Since ε and γ have arbitrary positive values, we deduce that $\lim_{k\to\infty} S_k^{1/k} = (\alpha+\beta)$. By Cauchy's criterion, this is equivalent to

$$\lim_{k\to\infty} \frac{S_k}{S_{k-1}} = (\alpha+\beta),$$

which ends the proof.

We can assume without loss of generality that, for $i = 1, \ldots, n$, $\beta_i \geq 1$. Otherwise, if there exists some $\beta_i \leq 1$, we consider the probability survival function at $s/\min_{i=1,\ldots,n}\{\beta_i\}$ instead of

s, which is equivalent to divide all the constants β_i by $\min_{i=1,\dots,n}\{\beta_i\}$.

We now prove by induction that

$$\lim_{k\to\infty} \frac{w(\Theta_{i,n},k)}{w(\Theta_{i,n},k-1)} = \sum_{i\in\Theta_{i,n}} \beta_i. \tag{2.8}$$

If $|\Theta_{i,n}| = 1$, e.g. $\Theta_{i,n} = \{j\}$, then we have

$$\lim_{k\to\infty} \frac{w(\Theta_{i,n},k)}{w(\Theta_{i,n},k-1)} = \lim_{k\to\infty} \frac{\alpha_j-(k-1)}{\alpha_j-k}\ \frac{\beta_j^k}{\beta_j^{k-1}} = \beta_j.$$

Assuming that the convergence (2.8) is true with $|\Theta_{i,n}| = j$. Let, for $i_{j+1} \in \Theta_n\backslash\Theta_{i,n}$, $\Theta_{i,n}^{(j+1)} = \Theta_{i,n} \cup \{i_{j+1}\}$. From the recursive algorithm for calculating the function w, we derive that

$$w(\Theta_{i,n}^{(j+1)},k) = \sum_{l=0}^{k} C_k^l \frac{\alpha_{i_{j+1}}\beta_{i_{j+1}}^l}{\alpha_{i_{j+1}}-l} w(\Theta_{i,n},k-l).$$

By using the previous proposition with

$$a_k = \frac{\alpha_{i_{j+1}}\beta_{i_{j+1}}^k}{\alpha_{i_{j+1}}-k}, \quad \alpha = \beta_{i_{j+1}}, \quad b_k = w(\Theta_{i,n},k), \quad \beta = \sum_{j\in\Theta_{i,n}} \beta_j,$$

we can deduce that

$$\lim_{k\to\infty} \frac{w(\Theta_{i,n}^{(j+1)},k)}{w(\Theta_{i,n}^{(j+1)},k-1)} = \sum_{l\in\Theta_{i,n}} \beta_l + \beta_{j+1} = \sum_{l\in\Theta_{i,n}^{(j+1)}} \beta_l.$$

Then, since

$$\lim_{k\to\infty} \frac{h(\Theta_{i,n},k)}{h(\Theta_{i,n},k-1)} = 1,$$

we derive that the coefficients of the power series expansion satisfy

$$\lim_{k\to\infty} \frac{h(\Theta_{i,n},k)w(\Theta_{i,n}^c,k)}{h(\Theta_{i,n},k-1)w(\Theta_{i,n}^c,k-1)} = \sum_{j\in\Theta_{i,n}^c} \beta_j = s_{i,n}.$$

Thus for large K, we have

$$
\begin{aligned}
&e_K(s) = \\
&(1+o(1))\left|\sum_{i=1}^{l_n}\prod_{j\in\Theta_i}\beta_j^{\alpha_j}c(\Theta_i)h(\Theta_i,K+1)w(\Theta_i^c,K+1)s^{-(\sum_{j\in\Theta_{i,n}}\alpha_j+K)}\left(1+\frac{s_{i,n}}{s}+\left(\frac{s_{i,n}}{s}\right)^2+\ldots\right)\right| \\
&\leq (1+o(1))\sum_{i=1}^{l_n}\left(\prod_{j\in\Theta_{i,n}}\beta_j^{\alpha_j}\right)\left|c(\Theta_{i,n})h(\Theta_{i,n},K+1)w(\Theta_{i,n}^c,K+1)\right|s^{-(\sum_{j\in\Theta_{i,n}}\alpha_j+K)}(s-s_{i,n})^{-1}.
\end{aligned}
$$

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Chapter 3

The sums of i.i.d. regularly varying random variables and simulation

This paper is concerned with the efficient simulation of $\mathbb{P}(S_n > s)$ in situations where s is large and S_n is the sum of n i.i.d. heavy-tailed random variables $X_1, \ldots, X_n$. The most efficient and simplest estimators introduced in the rare event simulation literature are those proposed by Asmussen and Kroese (2006) and Asmussen and Kortschak (2012). Although the main techniques for facing the rare event problem are importance sampling and splitting, the estimators of Asmussen, Kortschak and Kroese combine exchangeability arguments with conditional Monte-Carlo to construct estimators whose relative errors go to 0 as $s \to \infty$. In this paper, we decompose $\mathbb{P}(S_n > s)$ as the sum of $\mathbb{P}(M_n > s)$ and $\mathbb{P}(S_n > s, M_n < s)$ as proposed by Juneja (2007) because $\mathbb{P}(M_n > s)$ is known in closed form and is asymptotically equivalent to $\mathbb{P}(S_n > s)$. We construct new efficient estimators of $\mathbb{P}(S_n > s, M_n < s)$ by splitting up it again and then by using the same type of reliable methods as in Asmussen and Kroese (2006). We show that these new estimators have smaller relative errors than the estimators of Asmussen, Kortschak and Kroese. The conclusion of the numerical study is that our estimators compare extremely favorably with previous ones.

3.1 Introduction

In a probabilistic model, a rare event is an event with a very small probability of occurrence (typically between 10^{-6} and 10^{-10} or less). Rare events are of particular practical interest when dealing with systems where the rare event is a catastrophic failure with possible important human or monetary losses. Typical examples are given by failures in a public transport system or in a nuclear power plant, failures of information processing systems or telecommunication

networks, ruins of a large number of insurance companies or banks,... Therefore it is a critical issue to be able to evaluate the probability of these rare events.

In many cases, the mathematical models are often too complicated to be solved by analytical or numerical methods. Soft computing models and techniques are often used for complex systems that remain intractable to conventional mathematical and analytical methods. Among others we can quote artificial neural network models, image processing techniques, data-driven models coupled with data-preprocessing techniques, classifier ensemble methods, artificial neural network simulation methods (see e.g. for real applications: Cheng et al. (2005), Chau (2007), Huang and Chau (2008), Wu et al. (2009), Zhang and Chau (2009), Taormina et al. (2012)).

The simulation of the probabilistic models and the use of Monte Carlo methods provide more interesting alternative tools. However, estimation of rare event probabilities with the naive Monte Carlo techniques requires a prohibitively large number of trials. It is therefore necessary to use suitable variance reduction techniques as control variates, antithetic sampling, importance sampling, splitting,... In this paper, we focus on techniques for the efficient estimation of tail probabilities involving sums of heavy tailed random variables. It is amongst the simplest problems studied in the literature of rare event simulation, but also amongst the most challenging (see e.g. Chapter VI in Asmussen and Glynn (2007)). Such probabilities are of interest for insurance companies: the wealth of the company is modelled as a stochastic process that incorporates the gains due to insurance premiums and the losses due to claims. The failure event corresponds to the ruin of the company, i.e. when its wealth becomes negative.

More specifically, we consider non-negative, independent, identically distributed (i.i.d.) regularly varying heavy-tailed random variables $X_1, \ldots, X_n$ with common distribution F and let $S_n = X_1 + \ldots + X_n$. We are interested in the problem of efficiently estimating

$$z(s) = \mathbb{P}\left(S_n > s\right)$$

for large s. It is well-known that rare event simulation techniques are quite different in the light-tailed setting and in the heavy-tailed setting (see e.g. Chapter VI in Asmussen and Glynn (2007)). The simulation method we propose here is only concerned with regularly varying heavy-tailed distributions which are one of the most important subclasses of heavy-tailed distributions. However modifications of the algorithm would be needed for other classes of distributions.

Let us denote by $Z = Z(s)$ a random variable that can be generated by simulation and has expectation equal to $z(s)$. The usual performance measure is the relative error $e(s) = \left(\mathbb{V}ar\left(Z\right)\right)^{1/2} / z(s)$. An estimator has the logarithmically efficient property if $\limsup_{s\to\infty} z(s)^{\epsilon} e(s) < \infty$ for all $\varepsilon > 0$, it has the bounded relative error property if $\limsup_{s\to\infty} e(s) < \infty$ and it has the vanishing relative error property if $\lim_{s\to\infty} e(s) = 0$.

The first logarithmically efficient estimator was proposed by Asmussen and Binswanger (1997) and was based on a conditioning method. Asmussen, Binswanger and Hojgaard (2000) and Juneja and Shahabuddin (2002) used importance sampling techniques for estimating $z(s)$ and obtained logarithmically efficient estimators. Asmussen and Kroese (2006) proposed an estimator that combines an exchangeability argument with a conditional Monte Carlo idea

$$Z_{AK} = n\bar{F}\left(M_{n-1} \vee (s - S_{n-1})\right)$$

where $M_{n-1} = \max(X_1, \ldots, X_{n-1})$ and $S_{n-1} = X_1 + \ldots + X_{n-1}$. This estimator is shown in Asmussen and Kroese (2006) to have bounded relative error and in Hartinger and Kortschak (2009) to have vanishing relative error. The exact rates of decay of $e(s)$ for Z_{AK} have been recently given in Asmussen and Kortschak (2012) under the assumption that the probability density function of X_1 exists and is given by $f(x) = \alpha L(x)/x^{\alpha+1}$ where L is a slowly varying function. If $\alpha > 2$, they showed that, as $s \to \infty$,

$$e_{AK}(s) = (1 + o(1))\,\alpha\left[(n-1)\,\mathbb{V}ar(X_1)\right]^{1/2} s^{-1},$$

and if $\alpha < 2$, that, as $s \to \infty$,

$$e_{AK}(s) = (1 + o(1))\left[(n-1)\,k_\alpha \bar{F}(s)\right]^{1/2},$$

where k_α is a positive constant that depends on α (see Asmussen and Kortschak (2012) for the case $\alpha = 2$).

Asmussen and Kortschak (2012) also introduced related estimators with faster rates of decay for the two cases: $\alpha > 2$ and $1 < \alpha < 2$. If $\alpha > 2$, they suggested the following estimator

$$Z_{AKo^{(1)}} = Z_{AK} + n\left(\mathbb{E}\left[S_{n-1}\right] - S_{n-1}\right) f(s). \tag{3.1}$$

They gave its exact rates of decays when the first derivative of f exists and is given by $f'(x) = -\alpha(\alpha-1)L(x)/x^{\alpha+2}$. They showed that, if $\alpha > 4$, then

$$e_{AKo^{(1)}}(s) = (1 + o(1))\,\frac{\alpha(\alpha-1)}{2}\left[\mathbb{V}ar\left(S_{n-1}^2\right)\right]^{1/2} s^{-2},$$

as $s \to \infty$, and if $2 < \alpha < 4$, then

$$e_{AKo^{(1)}}(s) = (1 + o(1))\left[(n-1)\,k'_\alpha \bar{F}(s)\right]^{1/2},$$

as $s \to \infty$, where k'_α is a positive constant that depends on α (see Asmussen and Kortschak (2012) for the case $\alpha = 4$). If $1 < \alpha < 2$, they suggested to use an importance sampling method

to improve on Z_{AK} and proposed to consider the estimator $Z_{AKo}^{(2)} = Z^{(b)} + nZ^{(c)}$ where

$$\begin{aligned} Z^{(b)} &= n(n-1)\bar{F}(s/(2(n-1)))^2 I(S_n > s, X_{n-1} \wedge X_n \geq M_{n-2}) \\ Z^{(c)} &= \left(\bar{F}(s - S_{n-1}) - \bar{F}(s)\right) I\left(M_{n-1} \leq \frac{s}{2(n-1)}\right) R + \bar{F}(s)\,\mathbb{P}\left(M_{n-1} \leq \frac{s}{2(n-1)}\right) \end{aligned}$$

with $R = \prod_{i=1}^{n-1} f(X_i)/\tilde{f}(X_i)$ and $\tilde{f}$ an importance sampling density of the form $\tilde{L}(s)/s^{\tilde{\alpha}}$ such that $\tilde{\alpha} < 2\alpha - 2$. They showed that, as $s \to \infty$,

$$e_{AKo^{(2)}}(s) = O\left(s^{-1}\right).$$

Ghamami and Ross (2012) used the first time when the sum of the current maximum and S_n exceeds s to improve Z_{AK}: numerical examples for the standard Weibull distribution show that a variance reduction is gained despite a four time longer computer time, but no theoretical result on the rate of decay of the relative error is given. Other algorithms that share the feature of vanishing relative error have also been investigated but they could be more complicated to implement and their rates of decay are in general not very explicit: see Dupuis, Leder, and Wang (2007) and Hult and Svensson (2012) in the setting of dynamic importance sampling, and Gudmundsson and Hult (2012) in the setting of Markov chain Monte Carlo (MCMC).

In this paper, we will use the following decomposition

$$\mathbb{P}(S_n > s) = \mathbb{P}(M_n > s) + \mathbb{P}(S_n > s, M_n < s) \tag{3.2}$$

as in Juneja (2007). It is well-known that $\mathbb{P}(S_n > s) = (1 + o(1))\,\mathbb{P}(M_n > s)$, as $s \to \infty$, in the regularly varying case and that the probability $\mathbb{P}(M_n > s)$ is easily evaluated in closed form. Therefore, we only have to focus on developing efficient simulation techniques for $\mathbb{P}(S_n > s, M_n < s)$. Instead of considering the importance sampling technique (as in Juneja (2007)) that only leads to estimators with bounded relative errors, we rather continue to decompose the probability $\mathbb{P}(S_n > s, M_n < s)$:

$$\begin{aligned} \mathbb{P}(S_n > s) &= \mathbb{P}(M_n > s) + \mathbb{P}(M_n < s)\mathbb{P}(S_n > s | M_n < s) \\ &= \mathbb{P}(M_n > s) + \mathbb{P}(M_n < s) p_1(s) \end{aligned}$$

where

$$p_1(s) = \mathbb{P}(X_1^{[0,s]} + X_2^{[0,s]} + \cdots + X_n^{[0,s]} > s)$$

with $X_i^{[a,b]} = X_i | X_i \in [a,b]$ for any interval $[a,b] \subset \mathbb{R}^+$. We then define N_s as the number of random variables $X_i^{[0,s]}$ in the interval $[s/n, s]$. N_s has Binomial distribution with parameter

$$\mathbb{P}(X_i^{[0,s]} > s/n) = \frac{F(s) - F(s/n)}{F(s)} = (1 + o(1))\,(n^\alpha - 1)\bar{F}(s),$$

as $s \to \infty$. By the law of total probability, we have

$$\begin{aligned} p_1(s) &= \sum_{i=1}^{n} \mathbb{P}(N_s = i)\mathbb{P}(X_1^{[0,s]} + X_2^{[0,s]} + \cdots + X_n^{[0,s]} > s | N_s = i) \\ &= \sum_{i=1}^{n} \mathbb{P}(N_s = i) p_{1i}(s) \end{aligned}$$

where

$$p_{1i}(s) = \mathbb{P}(X_1^{[0,s/n]} + \cdots + X_{n-i}^{[0,s/n]} + X_{n-i+1}^{[s/n,s]} + \cdots + X_n^{[s/n,s]} > s).$$

Since $\mathbb{P}(M_n > s)$ and $\mathbb{P}(N_s = i)$ for $i = 1, \ldots, n$ are explicitly known, we focus on constructing independent random variables, P_{1i}, that can be generated by simulation and have expectation equal to $p_{1i}(s)$. Let

$$Z_{NR} = \mathbb{P}(M_n > s) + \mathbb{P}(M_n < s) \sum_{i=1}^{n} \mathbb{P}(N_s = i) P_{1i}.$$

The variance of this estimator of $z(s)$ is given by

$$\mathbb{V}ar(Z_{NR}) = \mathbb{P}(M_n < s)^2 \sum_{i=1}^{n} \mathbb{P}(N_s = i)^2 \mathbb{V}ar(P_{1i}).$$

Therefore more attention should be paid to P_{11} since $\mathbb{P}(N_s = i) = O(\bar{F}(s)^i)$. If we assume that, for $i = 2, \ldots, n$,

$$P_{1i} =_d I(X_1^{[0,s/n]} + \cdots + X_{n-i}^{[0,s/n]} + X_{n-i+1}^{[s/n,s]} + \cdots + X_n^{[s/n,s]})$$

then it is easely seen that, as $s \to \infty$,

$$\frac{\mathbb{V}ar(Z_{NR})}{\bar{F}(s)^2} = (1 + o(1))(n^\alpha - 1)^2 \mathbb{V}ar(P_{11}) + O(\bar{F}(s)^2).$$

The main contributions of the paper are: (i) to provide efficient estimators of $p_{11}(s)$ in order to derive new estimators of $\mathbb{P}(S_n > s)$ and (ii) to prove that these new estimators are more efficient than Z_{AK} and its improvements: $Z_{AKo^{(1)}}$ and $Z_{AKo^{(2)}}$.

Asmussen and Kortschak (2012) used S_{n-1} as a control variate with Z_{AK} for their estimator $Z_{AKo^{(1)}}$ (see Eq. (3.1)). We may note that, as mentioned in Section 3 of Juneja (2007), the method proposed in this paper may also be viewed as using $\mathbb{I}_{\{M_n > s\}}$ as a control variate in estimating the probability $\mathbb{P}(S_n > s)$ with control coefficient equal to 1. The control variates method is one of the most used variance reduction techniques for Monte Carlo methods. It is popular because of an effective variance reduction in a simple theoretical framework, but its main issue is the choice of a strong-correlated control variate as well as a suitable control

coefficient. We see that this technique has to be paired with conditional Monte Carlo in our case to provide finally efficient estimators.

This article is organised as follows. Section 2 introduces a first estimator that can be used for any regularly varying distribution with tail index $\alpha > 0$ such that the density function exists and is at least $I(\alpha)$ times differentiable where $I(\alpha)$ is the integer part of α. Section 3 introduces a second estimator that is less efficient that the previous estimator but more efficient that Z_{AK} for $0 < \alpha < 1$. This estimator is even though studied because it is not necessary to assume that the density function exists. For both estimators, we provide upper bounds for their relative errors.

3.2 A first improved estimator

We assume in this section that the probability density function of X_1 exists and is at least $I(\alpha)$ times differentiable for $\alpha \geq 1$. By the Taylor-Lagrange formula there exists $\mu_{s,x} \in [s-x, s]$ such that

$$f(s-x) = \sum_{k=0}^{I(\alpha)-1} (-1)^k f^{(k)}(s)\frac{x^k}{k!} + (-1)^{I(\alpha)} f^{(I(\alpha))}(\mu_{s,x})\frac{x^{I(\alpha)}}{I(\alpha)!}.$$

This type of regularity assumption appears to be quite standard when one wants to improve the efficiency of estimators. It is however crucial for this first approach.

Let $S^*_{n-1} = X_1^{[0,s/n]} + \cdots + X_{n-1}^{[0,s/n]}$. The probability $p_{11}(s)$ may be rewritten in the following way

$$\begin{aligned} p_{11}(s) &= \mathbb{P}(X_1^{[0,s/n]} + \cdots + X_{n-1}^{[0,s/n]} + X_n^{[s/n,s]} > s) \\ &= \frac{1}{F(s) - F(s/n)} \int_{s/n}^{s} f(x)\mathbb{P}(S^*_{n-1} > (s-x))dx \\ &= \frac{1}{F(s) - F(s/n)} \int_{0}^{(n-1)s/n} f(s-x)\mathbb{P}(S^*_{n-1} > x)dx. \end{aligned}$$

If $\alpha \geq 1$, we have

$$p_{11}(s) = \frac{1}{F(s) - F(s/n)} \left(\sum_{k=0}^{I(\alpha)-1} f^{(k)}(s)\frac{(-1)^k}{k!} M_{k,s} - \int_0^{(n-1)s/n} \Lambda(s,x)\mathbb{P}(S^*_{n-1} > x)dx \right)$$

where

$$\begin{aligned} M_{k,s} &= \int_0^{(n-1)s/n} x^k \mathbb{P}(S^*_{n-1} > x)dx = \frac{\mathbb{E}[(S^*_{n-1})^{(k+1)}]}{k+1} \\ &= \sum_{j_1+\ldots+j_{n-1}=k+1} \frac{k!}{j_1! \ldots j_{n-1}!} \mathbb{E}[(X_1^{[0,s/n]})^{j_1}] \ldots \mathbb{E}[(X_{n-1}^{[0,s/n]})^{j_{n-1}}] \end{aligned}$$

and

$$\Lambda(s,x) = \sum_{k=0}^{I(\alpha)-1} (-1)^k f^{(k)}(s)\frac{x^k}{k!} - f(s-x) = (-1)^{I(\alpha)+1} f^{(I(\alpha))}(\mu_{s,x})\frac{x^{I(\alpha)}}{I(\alpha)!}.$$

If $0 < \alpha < 1$, we let $\Lambda(s,x) = -f(s-x)$.

We denote by Y a random variable with uniform distribution on $[0,(n-1)/n]$. Then we have

$$\int_0^{(n-1)s/n} \Lambda(s,x)\mathbb{P}\left(S_{n-1}^* > x\right) dx = \frac{s\,(n-1)}{n}\mathbb{E}_Y\left[\Lambda(s,sY)\mathbb{P}(S_{n-1}^* > sY)\right].$$

By using an exchangeability argument as in Asmussen and Kroese (2006), we get

$$\begin{aligned} &\int_0^{(n-1)s/n} \Lambda(s,x)\mathbb{P}\left(S_{n-1}^* > x\right) dx \\ = \;& \frac{s\,(n-1)^2}{n}\mathbb{E}_{Y,X_1^{[0,s/n]},\cdots,X_{n-2}^{[0,s/n]}}\left[\Lambda(s,sY)\bar{F}_{X_{n-1}^{[0,s/n]}}\left(M_{n-2}^{\times}\wedge(s/n)\right)\right] \end{aligned}$$

where

$$M_{n-2}^{\times} =_d \max(X_1^{[0,s/n]},\cdots,X_{n-2}^{[0,s/n]}, sY - X_1^{[0,s/n]} - \cdots - X_{n-2}^{[0,s/n]})$$

and

$$\bar{F}_{X_{n-1}^{[0,s/n]}}(x) = \left[\bar{F}(x) - \bar{F}(s/n)\right]/F(s/n).$$

Therefore we propose the following first estimator for $p_{11}(s)$

$$P_{11}^{(1)} = \frac{1}{\bar{F}(s/n) - \bar{F}(s)}\left(\sum_{k=0}^{I(\alpha)-1} f^{(k)}(s)\frac{(-1)^k}{k!}M_{k,s} - \frac{s\,(n-1)^2}{n}\Lambda(s,sY)\frac{\bar{F}(M_{n-2}^{\times}\wedge(s/n)) - \bar{F}(s/n)}{F(s/n)}\right).$$

and the first estimator for $z(s)$

$$Z_{NR^{(1)}} = \mathbb{P}(M_n > s) + \mathbb{P}(M_n < s)\left(\mathbb{P}(N_s = 1)P_{11}^{(1)} + \sum_{i=2}^{n}\mathbb{P}(N_s = i)P_{1i}\right).$$

Theorem 3.1. *Let $R(\alpha) = \alpha - I(\alpha)$. Then if $R(\alpha) < 1/2$*

$$e_{NR^{(1)}}(s) = O\left(\bar{F}(s)\right)$$

as $s \to \infty$, and if $R(\alpha) > 1/2$

$$e_{NR^{(1)}}(s) = O\left(s^{-1/2-I(\alpha)}\right)$$

as $s \to \infty$.

Note that, if f was only p times differentiable with $p \in \mathbb{N}$ and $0 \le p < I(\alpha)$, we would obtain $e_{NR^{(1)}}(s) \le O\left(s^{-1/2-p}\right)$.

Proof: We have

$$\begin{aligned}\mathbb{V}ar\left(P_{11}^{(1)}\right) &= \frac{\left(\bar{F}(s/n)\right)^2}{(\bar{F}(s/n)-\bar{F}(s))^2}\left(\frac{s\,(n-1)^2}{F(s/n)n}\right)^2 \mathbb{V}ar\left(\Lambda(s,sY)\left(\frac{\bar{F}(M_{n-2}^{\times}\wedge(s/n))}{\bar{F}(s/n)}-1\right)\right)\\ &\leq \frac{\left(\bar{F}(s/n)\right)^2}{(\bar{F}(s/n)-\bar{F}(s))^2}\left(\frac{s\,(n-1)^2}{F(s/n)n}\right)^2 \mathbb{E}\left[\Lambda(s,sY)^2\left(\frac{\bar{F}(M_{n-2}^{\times}\wedge(s/n))}{\bar{F}(s/n)}-1\right)^2\right].\end{aligned}$$

Note that $\mu_{s,sY}\in[s(1-Y),s]\subset[s/n,s]$ and that

$$\sup_{s\geq 0}\sup_{s/n\leq x\leq s}\frac{f^{(I(\alpha))}(x)}{f^{(I(\alpha))}(s)}<\infty$$

since $f^{(I(\alpha))}$ is a regularly varying function. Therefore

$$\Lambda(s,sY)^2=\left(\frac{f^{(I(\alpha))}(\mu_{s,sY})}{I(\alpha)!}\right)^2(sY)^{2I(\alpha)}=O\left(\left[L(s)s^{-(\alpha+1)}\right]^2\right)Y^{2I(\alpha)}$$

as $s\to\infty$.

Then, using the same arguments as in Asmussen and Glynn (2007) p. 275, we deduce that $M_{n-2}^{\times}$ is larger than $sY/(n-1)$. It follows that

$$\bar{F}(s/n)\leq\bar{F}(M_{n-2}^{\times}\wedge(s/n))\leq\bar{F}(sY/(n-1))$$

and we get

$$\mathbb{V}ar\left(P_{11}^{(1)}\right)=O\left(\bar{F}(s)^2\right)\mathbb{E}\left[Y^{2I(\alpha)}\left(\frac{\bar{F}(sY/(n-1))}{\bar{F}(s/n)}-1\right)^2\right].$$

If U has a uniform distribution on $[0,1]$, we may write

$$\begin{aligned}\mathbb{V}ar\left(P_{11}^{(1)}\right) &= O\left(\bar{F}(s)^2\right)\mathbb{E}\left[U^{2I(\alpha)}\left(\frac{\bar{F}(sU/n)}{\bar{F}(s/n)}-1\right)^2\right]\\ &= O\left(\bar{F}(s)^2\right)\mathbb{E}\left[U^{2I(\alpha)}\left(\left(\frac{\bar{F}(sU)}{\bar{F}(s)}\right)^2-2\frac{\bar{F}(sU)}{\bar{F}(s)}+1\right)\right].\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}\left[U^{2I(\alpha)}\frac{\bar{F}(sU)}{\bar{F}(s)}\right] &= \frac{1}{\bar{F}(s)}\int_0^1 u^{2I(\alpha)}\bar{F}(su)du=\frac{s^{-1-2I(\alpha)}}{\bar{F}(s)}\int_0^s v^{2I(\alpha)}\bar{F}(v)dv\\ &= \frac{s^{-1-2I(\alpha)}}{\bar{F}(s)}\int_0^s v^{2I(\alpha)-\alpha}L(v)dv.\end{aligned}$$

Since $2I(\alpha)-\alpha>-1$ for $\alpha>0$, we deduce from Karamata theorem that, as $s\to\infty$,

$$\int_0^s v^{2I(\alpha)-\alpha}L(v)dv=\frac{(1+o(1))}{2I(\alpha)-\alpha+1}s^{2I(\alpha)-\alpha+1}L(s)$$

and

$$\mathbb{E}\left[U^{2I(\alpha)}\frac{\bar{F}(sU)}{\bar{F}(s)}\right]=\frac{(1+o(1))}{2I(\alpha)-\alpha+1}.$$

Moreover

$$\begin{aligned}\mathbb{E}\left[U^{2I(\alpha)}\left(\frac{\bar{F}(sU)}{\bar{F}(s)}\right)^2\right] &= \frac{1}{\bar{F}(s)^2}\int_0^1 u^{2I(\alpha)}\bar{F}^2(su)du=\frac{s^{-1-2I(\alpha)}}{\bar{F}^2(s)}\int_0^s v^{2I(\alpha)}\bar{F}^2(v)dv\\ &= \frac{s^{-1-2I(\alpha)}}{\bar{F}^2(s)}\int_0^s v^{2I(\alpha)-2\alpha}L^2(v)dv.\end{aligned}$$

When $I(\alpha)-\alpha>-1/2$ or equivalenty $R(\alpha)<1/2$, we deduce from Karamata theorem that, as $s\to\infty$,

$$\mathbb{E}\left[U^{2I(\alpha)}\left(\frac{\bar{F}(sU)}{\bar{F}(s)}\right)^2\right]=\frac{(1+o(1))}{2I(\alpha)-2\alpha+1}.$$

When $I(\alpha)-\alpha<-1/2$ or equivalenty $R(\alpha)>1/2$, we have, as $s\to\infty$,

$$\mathbb{E}\left[U^{2I(\alpha)}\left(\frac{\bar{F}(sU)}{\bar{F}(s)}\right)^2\right]=(1+o(1))\,\frac{s^{-1-2I(\alpha)}}{\bar{F}^2(s)}.$$

Therefore, if $R(\alpha)<1/2$, $e_{NR}(s)=O\left(\bar{F}\left(s\right)\right)$ and, if $R(\alpha)>1/2$, $e_{NR}(s)=O\left(s^{-1/2-I(\alpha)}\right)$.

3.3 Second improved estimator

We now propose and study a second estimator for which it is not necessary to assume that the density function exists. We will show that it is less efficient that the previous estimator $Z_{NR^{(1)}}$ but it is more efficient that Z_{AK} for $0<\alpha<1$.

For $s>\left((n-1)/n\right)^{-1-1/\alpha}$, we split up $p_{11}(s)$ into two parts

$$\begin{aligned}p_{11}(s) &= \mathbb{P}(X_1^{[0,s/n]}+\cdots+X_{n-1}^{[0,s/n]}+X_n^{[s/n,s]}>s)\\ &= \mathbb{P}(X_n^{[s/n,s]}<s-s^{1/(1+\alpha)})p_{11d}(s)+\mathbb{P}(X_n^{[s/n,s]}>s-s^{1/(1+\alpha)})p_{11u}(s)\end{aligned}$$

where

$$\begin{aligned}p_{11d}(s) &= \mathbb{P}(X_1^{[0,s/n]}+\cdots+X_{n-1}^{[0,s/n]}+X_n^{[s/n,s-s^{1/(1+\alpha)}]}>s)\\ p_{11u}(s) &= \mathbb{P}(X_1^{[0,s/n]}+\cdots+X_{n-1}^{[0,s/n]}+X_n^{[s-s^{1/(1+\alpha)},s]}>s).\end{aligned}$$

We then use the method of Asmussen and Kroese (2006) to estimate $p_{11d}(s)$ and $p_{11u}(s)$ respectively by

$$P_{11d} = (n-1)\,\bar{F}_{X_{n-1}^{[0,s/n]}}(M_{n-2}^{\times,d}) \quad \text{and} \quad P_{11u} = (n-1)\,\bar{F}_{X_{n-1}^{[0,s/n]}}(M_{n-2}^{\times,u}).$$

where $M_{n-2,d}^{\times}$ and $M_{n-2,u}^{\times}$ are two independent random variables such that

$$\begin{aligned} M_{n-2,d}^{\times} &=_d \max(X_1^{[0,s/n]}, \cdots, X_{n-2}^{[0,s/n]}, s - X_n^{[s/n,s-s^{1/(1+\alpha)}]} - X_1^{[0,s/n]} + \cdots + X_{n-2}^{[0,s/n]}) \\ M_{n-2,u}^{\times} &=_d \max(X_1^{[0,s/n]}, \cdots, X_{n-2}^{[0,s/n]}, s - X_n^{[s-s^{1/(1+\alpha)},s]} - X_1^{[0,s/n]} + \cdots + X_{n-2}^{[0,s/n]}). \end{aligned}$$

Therefore we propose the following second estimator for $p_{11}(s)$

$$P_{11}^{(2)} = (n-1)\,\mathbb{P}(X_n^{[s/n,s]} < s - s^{1/(1+\alpha)})\bar{F}_{X_{n-1}^{[0,s/n]}}(M_{n-2}^{\times,d}) + (n-1)\,\mathbb{P}(X_n^{[s/n,s]} > s - s^{1/(1+\alpha)})\bar{F}_{X_{n-1}^{[0,s/n]}}(M_{n-2}^{\times,u}).$$

and the second estimator for $z(s)$

$$Z_{NR^{(2)}} = \mathbb{P}(M_n > s) + \mathbb{P}(M_n < s)\left(\mathbb{P}(N_s = 1)P_{11}^{(2)} + \sum_{i=2}^{n}\mathbb{P}(N_s = i)P_{1i}\right).$$

The threshold $s - s^{1/(1+\alpha)}$ has been chosen such that the variances of the two components of $P_{11}^{(2)}$ contribute to the total variance in the same way. The following theorem gives an asymptotic upper bound of the relative error of $Z_{NR^{(2)}}$.

Theorem 3.2. *We have*

$$e_{NR^{(2)}}(s) = O\left(s^{-\alpha/(1+\alpha)}\right)$$

as $s \to \infty$.

Proof: We have

$$\mathbb{V}ar\left(P_{11}^{(2)}\right) = \mathbb{P}(X_n^{[s/n,s]} < s - s^{1/(1+\alpha)})^2\mathbb{V}ar\left(P_{11d}\right) + \mathbb{P}(X_n^{[s/n,s]} > s - s^{1/(1+\alpha)})^2\mathbb{V}ar\left(P_{11u}\right).$$

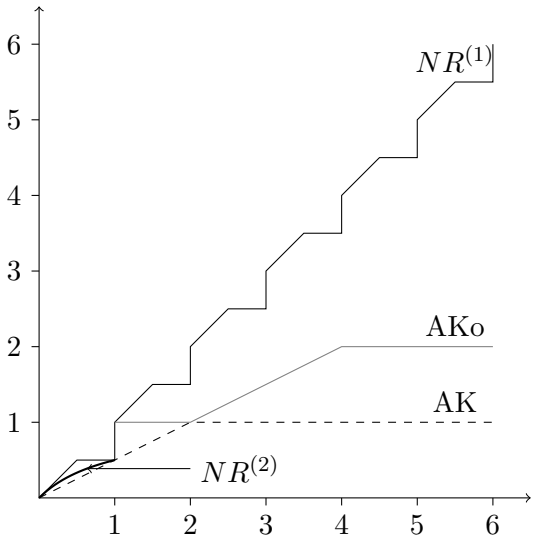
Using the same arguments as in Asmussen and Glynn (2007) p. 275, we deduce that

$$M_{n-2,d}^{\times} \geq \frac{s - X_n^{[s/n,s-s^{1/(1+\alpha)}]}}{n-1} \geq \frac{s^{1/(1+\alpha)}}{n-1}.$$

Therefore we have

$$\begin{aligned} \mathbb{V}ar\left(P_{11d}\right) &\leq (n-1)^2\,\bar{F}^2_{X_{n-1}^{[0,s/n]}}\left(\frac{s^{1/(1+\alpha)}}{n-1}\right) = (n-1)^2\left(\frac{\bar{F}(s^{1/(1+\alpha)}/(n-1)) - \bar{F}(s/n)}{F(s/n)}\right)^2 \\ &= O\left(\bar{F}(s^{1/(1+\alpha)})^2\right) \end{aligned}$$

FIGURE 3.1: Measures of efficiency of the estimators



as $s \to \infty$. Now note that

$$\begin{aligned}\mathbb{P}\left(X_n^{[s/n,s]} > s - s^{1/(1+\alpha)}\right) &= \frac{F(s) - F(s - s^{1/(1+\alpha)})}{F(s) - F(s/n)} = \frac{\bar{F}(s - s^{1/(1+\alpha)}) - \bar{F}(s)}{\bar{F}(s/n) - \bar{F}(s)} \\ &= \frac{L\left(s\left(1 - s^{-\alpha/(1+\alpha)}\right)\right)}{L(s)} O(s^{-\alpha/(1+\alpha)}) = O(s^{-\alpha/(1+\alpha)}).\end{aligned}$$

Moreover its is clear that

$$\mathbb{P}(X_n^{[s/n,s]} < s - s^{1/(1+\alpha)}) = O(1) \quad \text{and} \quad \mathbb{V}ar\,(P_{11u}) = O(1).$$

It follows that

$$\mathbb{V}ar\,(P_{11}) = O\left(s^{-2\alpha/(1+\alpha)}\right)$$

and $e_{NR}(s) = O\left(s^{-\alpha/(1+\alpha)}\right)$.

3.4 Comparisons of the relative errors and numerical examples

For the comparisons of the estimators, we assume without loss of generality that the slowly varying function of F is equal to a constant. We then define for an estimator Z of $z(s)$ a measure of its efficiency by

$$\gamma(Z) = \liminf_{s\to\infty}\left(-\frac{\log e(s)}{\log s}\right).$$

The larger this measure, the faster is the rate of decay of the relative error of the estimator.

Figure 3.1 gives the exact values of γ for the estimators Z_{AK} ($\alpha > 0$), $Z_{AKo^{(1)}}$ ($\alpha > 2$), but only lower bounds for the estimators $Z_{AKo^{(2)}}$ ($1 < \alpha < 2$), $Z_{NR^{(1)}}$ ($\alpha > 0$) and $Z_{NR^{(2)}}$ ($0 < \alpha < 1$) as presented in the previous sections.

We see that the proposed estimators ($Z_{NR^{(1)}}$ and $Z_{NR^{(2)}}$) provide from a theoretical point of view improvements of Asmussen-Kroese estimator (Z_{AK}) and Asmussen-Kortschack estimators ($Z_{AKo^{(1)}}$ and $Z_{AKo^{(2)}}$). We performed numerical experiments to check that these improvements may be substantial in practice. We considered the Lomax distribution, also called Pareto Type II distributions, for the distribution of X_i

$$\mathbb{P}(X_i > x) = \left(1 + \frac{x}{\beta}\right)^{-\alpha}, \qquad x \geq 0.$$

It is essentially a Pareto distribution that has been shifted so that its support begins at zero. We provide in Table 3.1 numerical results for the standard Lomax distribution ($\beta = 1$) for several values of α (between 0.3 and 4.5) and s. The number of terms in the sum is fixed to $n = 5$. The exact values of $z(s)$ are computed by using the approach developed in Nguyen and Robert (2013). The choice of this distribution and the choices of the values of its parameters are quite standard for comparisons of different simulation methods in the regularly varying case (see e.g. Asmussen and Kroese (2006), Hult and Svensson (2012)). We performed other simulations for other sets of parameters ($\beta = 0.5, 2, 5$ and $n = 10, 30, 50$). Qualitatively similar results were found.

Table 3.1 indicates that a substantial variance reduction is gained with $Z_{NR^{(1)}}$ for all values of α. $Z_{NR^{(2)}}$ is less efficient than $Z_{NR^{(1)}}$ but also provides very good results when $0 < \alpha < 1$.

3.5 Conclusion

This paper presents new algorithms to estimate efficiently the probability $\mathbb{P}(S_n > s)$ when s is large and S_n is the sum of n i.i.d. regularly varying random variables. The numerical study shows that the new estimators compare extremely favorably with previous ones. It is, of course, recognized that there are limitations in the assumptions used in this study (in particular for the first improved estimator for which it is assumed that the probability density function of the random variables is sufficiently differentiable). In future, we will focus on how to take into account not identically distributed random variables in the sum or how to take into account extreme dependence between the random variables. In this way, the applicability of the algorithms will be hugely improved.

Note however that, by using additional higher order correction terms for $Z_{AKo^{(1)}}$, it could be possible to improve it.

TABLE 3.1: Comparisons of the relative errors of the estimators

α	s	z	$e_{Z_{AK}}$	$e_{Z_{AKo}}$	$e_{Z_{NR^{(1)}}}$	$e_{Z_{NR^{(2)}}}$
0.3	10^7	0.34171	0.37066		0.12173	0.12170
0.3	10^10	0.04909	0.13014		0.02450	0.02462
0.5	10^4	0.04997	0.12747		0.01789	0.02073
0.5	10^6	0.00499	0.03997		0.001864	0.003141
0.7	10^4	0.08625	0.22165		0.59683	0.60577
0.7	10^7	6.299e-04	0.02316		0.007010	0.012578
0.9	10^4	1.265e-03	0.03152		0.003580	0.01067
0.9	10^5	1.582e-04	0.01740		0.000467	0.00193
1.5	1000	1.833e-03	0.13152		0.020663	
1.5	10^4	5.073e-05	0.01384		0.007161	
1.5	5×10^4	4.484e-06	0.06678		0.046846	
2.5	500	1.165e-05	0.07388	0.03128	8.521e-04	
2.5	2000	2.907e-07	0.01506	1.86e-03	1.611e-04	
2.5	5000	2.873e-08	0.00565	1.58e-03	1.266e-05	
3.5	200	5.832e-07	0.08896	0.04663	6.059e-03	
3.5	2000	1.444e-10	7.54e-03	4.04e-04	1.730e-06	
4.5	100	7.750e-08	0.14966	0.09128	0.010299	
4.5	1000	1.679e-12	0.01246	0.00136	4.19e-07	

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Chapter 4

The sums of dependent regularly varying variables and simulation

This paper focuses on efficient simulation of $z(s) = \mathbb{P}(S_n > s)$ in the situation where s is a large value and S_n is the sum of n heavy-tailed random variables with the assumption that the dependence is either an Archimedean copula or an Archimedean survival copula. In recent research on this problem, almost all authors paid attention to the calculation of limit $\lim_{s\to\infty} \frac{P(S_n>s)}{P(X_1>s)}$ where the distribution functions of X_i; $i = 1, \cdots, n$ are identical. Obviously, these results can not be applied if the marginal random variables are differently distributed. Moreover, the rate of convergence is hard to verify. Thus, we will estimate the probability $z(s)$ using different simulation technique approaches. Simulation method has the advantage that it can be applied for any range of dependence without identical marginal variables assumption. Numerical studies are performed with different level of s and several Archimedean copulas in Section 4.4 prove that our estimators are efficient at calculating probability $z(s)$

4.1 Introduction

Rare event simulation is a technique of simulating a small or very small probability (typically between 10^{-6} and 10^{-10} or less). Generally, the analytical and numerical methods are impossible or too complicated to calculate such a small probability, especially in the case of dependence. On the other hand, simulation techniques can be applied in complex situations such as high dimensional calculation or the calculation with assumption of dependence between marginal variables. However, the classical Monte Carlo method does not work on small probability simulation since the relative errors (variation coefficient) are too large. So in rare event simulation,

variance reduction techniques are key solutions. To evaluate an estimator in such context, Asmussen and Glynn [3] introduced relative error as a measure of efficiency of an estimator, which is defined as follows.

Definition 4.1 (Asmussen and Glynn, 2006)**.** *An unbiased estimator $Z(s)$ of probability $z(s) = \mathbb{P}(S_n > s)$ whose relative error calculated as follows $e(Z(s)) = \frac{\mathbb{S}d(Z(s))}{\mathbb{E}(Z(s))}$, is called*

- *Logarithmically efficient estimator if* $\limsup_{s\to\infty} e(Z(s))\ [z(s)]^{\epsilon} = 0$ *for all* $\epsilon > 0$.
- *Estimator with bounded relative error if* $\limsup_{s\to\infty} e(Z(s)) < \infty$.
- *Estimator with vanishing relative error* $\limsup_{s\to\infty} e(Z(s)) = 0$.

In fact, estimating tail distribution of the sums of dependent random variables via simulation is challenging. It requires a specific expression of dependence structure or a closed form of conditional distribution functions, the case of elliptic distributions is an example. When the dependence structure is elliptic, Blanchet and Rojas-Nandayapa [8] proposed a conditional Monte Carlo estimator for tail distribution of the sum of log-elliptic random variables and proved that it has a logarithmically efficient relative error. The sum of the log-elliptic random variables was also estimated by Kortschak and Hashorva [13] using the simulation method that Asmussen and Kroese introduced in [5], which presents favorable results especially in multivariate log-normal case.

This paper considers the dependence structure of Archimedean copula and Archimedean survival copula. Having been defined by Nelsen [15], an n dimensional Archimedean copula C of generator Φ is multivariate distribution function of n uniform random variables $(U_1, U_2, \cdots, U_n)$

$$\begin{aligned} C(u_1, u_2, \cdots, u_n) &= \mathbb{P}(U_1 \le u_1, U_2 \le u_2, \cdots, U_n \le u_n) \\ &= \Phi\big(\Phi^{\leftarrow}(u_1) + \Phi^{\leftarrow}(u_2) + \cdots + \Phi^{\leftarrow}(u_n)\big) \end{aligned}$$

where $\Phi^{\leftarrow}$ is the inverse function of Φ. The inverse function of a function g is defined by

$$g^{\leftarrow}(x) = \inf\{t \in \mathbb{R}^+ : g(t) \ge x\}.$$

The conditions for Φ such that C is a multivariate distribution function is mentioned in McNeil et al. [14] and will be discussed in Section 4.2.

Definition 4.2. *Survival copula of $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$, denoted $\tilde{C}$, is the copula associated with the survival function of $\boldsymbol{X}$:*

$$\tilde{C}\left(\bar{F}_1(x_1), \cdots, \bar{F}_n(x_n)\right) = \bar{F}\left(x_1, x_2, \cdots, x_n\right)$$

In particular, if C is the copula of $\boldsymbol{X}$ and let $\boldsymbol{U} = (U_1, U_2, \cdots, U_d)$ be a vector such that $\boldsymbol{U}$ follows C, then we have the relation:

$$\tilde{C}(u_1, \cdots, u_n) = \bar{C}\left(1 - u_1, \cdots, 1 - u_n\right)$$

where $\bar{C}(\boldsymbol{u}) = \mathbb{P}(U_1 > u_1, \cdots, U_n > u_n)$.

Obviously, there is an one-one mapping from the set of Archimedean copulas and the set of Archimedean survival copulas. In this paper, the notation *"Archimedean survival copula C of generator Φ"* is used to mention the survival copula of vector $\tilde{\mathbf{U}}$ is the Archimedean copula of generator Φ.

Under the dependence structure of Archimedean copula C, we denote n marginal random variables as $X_1, X_2, \cdots, X_n$ where the distribution functions of X_i are F_i for $i = 1, 2, \cdots, n$. If vector $\mathbf{U} = (U_1, U_2, \cdots, U_n)$ follows copula C where U_i are uniformly distributed in $(0, 1)$ for $i = 1, 2, \cdots, n$, then we have the stochastic representation for vector $\mathbf{X}$:

$$(X_1, X_2, \cdots, X_n) \stackrel{d}{=} (F_1^{\leftarrow}(U_1), F_2^{\leftarrow}(U_2), \cdots, F_n^{\leftarrow}(U_n)).$$

The multivariate distribution function of $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ as a consequence is

$$\begin{aligned} \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) &= \mathbb{P}(F_1(X_1) \leq F_1(x_1), \cdots, F_n(X_n) \leq F_n(x_n)) \\ &= C(F_1(x_1), F_2(x_2), \cdots, F_n(x_n)). \end{aligned}$$

Under the dependence structure of Archimedean survival copula C, we denote the marginal variables as $Y_1, Y_2, \cdots, Y_n$. The i^{th} marginal variable Y_i has the same distribution function F_i with X_i and there exists a unique vector $\mathbf{U}$ of uniform marginal distribution following an Archimedean copula C such that

$$(Y_1, Y_2, \cdots, Y_n) \stackrel{d}{=} \left(\bar{F}_1^{\leftarrow}(U_1), \bar{F}_2^{\leftarrow}(U_2), \cdots, \bar{F}_n^{\leftarrow}(U_n)\right).$$

The multivariate survival distribution function of $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$ can be written via copula C as follows

$$\begin{aligned} \mathbb{P}(Y_1 > y_1, Y_2 > y_2, \cdots, Y_n > Y_n) &= \mathbb{P}(\bar{F}_1(Y_1) < \bar{F}_1(y_1), \cdots, \bar{F}_n(Y_n) < \bar{F}_n(y_n)) \\ &= C\left(\bar{F}_1(y_1), \bar{F}_2(y_2), \cdots, \bar{F}_n(y_n)\right). \end{aligned}$$

Assumption of marginal distributions

In this paper, the marginal random variables are assumed to be positive regularly varying, i.e the marginal survival distributions $\bar{F}_1, \bar{F}_2, \cdots, \bar{F}_n$ are regularly varying functions.

Definition 4.3. *A random variable X of survival distribution function $\bar{F}_X = \mathbb{P}(X > x)$ is said to be regularly varying at infinite with index $-\alpha$ with $\alpha \geq 0$ (written $\bar{F} \in \mathcal{RV}_\infty(-\alpha)$) if*

$$\lim_{x\to\infty} \frac{\bar{F}_X(tx)}{\bar{F}_X(x)} = t^{-\alpha}$$

In the case of $\alpha = 0$, function $\bar{F}_X$ is said to be slowly varying.

With the assumption of regular variation, we denote the tail indices of $\bar{F}_1, \bar{F}_2, \cdots, \bar{F}_n$ by $\alpha_1, \alpha_2, \cdots, \alpha_n$ respectively. Note that the marginals are not necessarily identical. By the definition, we can write $\bar{F}_i(x) = l_i(x)x^{-\alpha_i}$ where l_i are different slowly varying functions and α_i satisfy $\alpha_1 \leq \alpha_2 \cdots \leq \alpha_n$. It is well-known that if all the variables are positive, then we have the following upper and lower bounds for $z(s)$

$$\begin{aligned} & \max_{i=1,2\cdots,n} \mathbb{P}(X_i > s) < \mathbb{P}(S_n > s) \leq \sum_{i=1}^{n} \mathbb{P}(X_i > s/n) \\ \rightarrow \quad & \max_{i=1,2\cdots,n} \bar{F}_i(s) < z(s) \leq n \max_{i=1,2\cdots,n} \bar{F}_i(s/n). \end{aligned}$$

Obviously, if α_1 is strictly less than α_2, then the upper and the lower bounds become asymptotically:

$$\bar{F}_1(s) < z(s) \leq n\bar{F}_1(s/n).$$

Asymptotic result for the sums of regularly varying variables

There are a number of results for the sum of dependent regularly varying random variables focusing on the asymptotic behavior of tail distribution of the sum. When marginal variables are tail independent i.e $\lim_{x_i \wedge x_j \to \infty} \mathbb{P}(X_i > x_i | X_j > x_j) = 0$ for all $i \neq j$, Jessen and Mikosch [11] showed that if there exist $n-1$ constants $c_2^+, \cdots, c_n^+$ such that $c_i^+ = \lim_{s\to\infty} \frac{\bar{F}_i(s)}{\bar{F}_1(s)}$, then

$$\lim_{s\to\infty} \frac{\mathbb{P}(S_n > s)}{\bar{F}_1(s)} = 1 + \sum_{i=2}^{n} c_i^+.$$

If the marginal distributions are i.i.d, we have ($c_i^+ = 1 \ \forall i$) which leads to a well-known result $z(s) \sim n\bar{F}_1(s)$. The sum of asymptotic independent regularly varying variable was also studied by Yuen and Yin [19]. The authors proposed a result with weaker assumption than Jessen and Mikosch [11] in the case of n is deterministic and then developed this result in the case of n is a random variable (see Yuen and Yin [19]). A research for the sum of dependent and identical regularly varying variables is carried out in Albrecher et al. [2]. They analyzed the tail behavior of the sum of two regularly random variables following the coefficient $\hat{\lambda} = \lim_{s\to\infty} \mathbb{P}(X_1 > s | X_2 > s)$

and the ratio $c = \lim_{s\to\infty} \frac{\mathbb{P}(X_1>s)}{\mathbb{P}(X_2>s)}$. Moreover, in Albrecher et al.'s [2], if the dependence is a bivariate copula, the authors presented an asymptotic result for the sum via the density copula (see [2]).

Under the assumption of Archimedean survival copula dependence structure, Wuthrich [18] obtained the limit $q_n(\alpha,\beta) = \frac{P(S_n^Y>s)}{P(Y_1>s)}$ with additional assumptions that marginal distributions are identical and Archimedean generator is a regularly varying function.

Proposition 4.4 (Wuthrich, 2003)**.** *Suppose that $Y_1, Y_2, \cdots, Y_n$ are n regularly varying random variables of common survival distribution function $\bar{F}_Y \in \mathcal{RV}_\infty(-\alpha), \alpha > 0$; and the dependence between $Y_i; i = 1, 2, \cdots, n$ is an Archimedean survival copula of generator Φ. If Φ is a regularly varying function i.e $\Phi \in \mathcal{RV}_\infty(-\beta), \beta > 0$, then there exists the limit $q_n(\alpha,\beta)$:*

$$\lim_{s\to\infty} \frac{\mathbb{P}(Y_1 + \cdots + Y_n)}{\mathbb{P}(Y_1 > s)} = \lim_{s\to\infty} \frac{\mathbb{P}(S_n^Y > s)}{\bar{F}_Y(s)} = q_n(\alpha,\beta).$$

This limit can be calculated by a recursive formula:

$$q_n(\alpha,\beta) = \sum_{i=1}^{n} (-1)^{(i-1)}\ C_n^i\ q_{n,i}(\alpha,\beta)\ i^{-\beta}$$

where the $q_{n,i}(\alpha,\beta)$ is the probability of

$$q_{n,i}(\alpha,\beta) = \mathbb{P}\big(1/H_1^{(i)} + \cdots + 1/H_n^{(i)} \geq n\big).$$

In this probability, vector $(H_1^{(i)}, \cdots, H_n^{(i)})$ has multivariate distribution function $\mathcal{H}_{n,i}^{(\alpha,\beta)}$ determined by

$$\mathcal{H}_{n,i}^{(\alpha,\beta)}(h_1, \cdots, h_n) = \left(\frac{h_1^{-\alpha/\beta} + \cdots + h_n^{-\alpha/\beta}}{i}\right)^{-\beta}$$

for $(h_1, \cdots, h_n) \in [0,1]^i \times [0,\infty)^{n-i}$.

Sun and Li [17] examined the relation between multivariate regularly varying vector and Archimedean copula with regularly varying generator to acquire an integral form for $q_n(\alpha,\beta)$. In their research, the marginal distributions are assumed to be identical and regularly varying. In the case of Archimedean copula, if the generator satisfies $(1-\Phi) \in \mathcal{RV}_0(-\beta)$, the following limit exists

$$\lim_{s\to\infty} \frac{P(S_n^X > s)}{P(X_1 > s)} = q_n^C(\alpha,\beta)$$

where

$$q_n^C(\alpha,\beta) = \int^{\sum v_i^{-1/\alpha}>1} \frac{\delta^n}{\delta v_1 \cdots \delta v_n} \sum^{1\le i_1,\cdots,i_j\le n} \left[(-1)^{j-1}(v_{i_1}^{1/\beta} + \cdots + v_{i_j}^{1/\beta})^{\beta}\right] dv_1 \cdots dv_n.$$

In the case of Archimedean survival copula, if the generator satisfies $\Phi \in \mathcal{RV}_\infty(-\beta)$, then

$$\lim_{s\to\infty} \frac{P(S_n^Y > s)}{P(Y_1 > s)} = q_n^C(\alpha,\beta)$$

where

$$q_n^C(\alpha,\beta) = \int^{\sum v_i^{-1}>1} \frac{\delta^n}{\delta v_1 \cdots \delta v_n} \left(v_1^{-\alpha/\beta} + \cdots + v_n^{-\alpha/\beta}\right)^{-\beta} dv_1 \cdots dv_n.$$

In this paper, we estimate the probability $z(s)$ using techniques of conditional Monte Carlo simulation. The advantage is that we can reduce the complexity in calculating tail distribution of the sum even if marginal distributions are different. Moreover, we can improve the performance by increasing the number of replications. Our research focuses on the dependence structure of Archimedean copula and Archimedean survival copula. In the following sections, we introduce some simulation techniques related to Archimedean copula which are used to develop our estimators.

4.2 Archimedean copula and simulation

The classical simulation method for a dependent vector bases on conditional distributions. Supposed that vector $\mathbf{U} = (U_1, U_2, \cdots, U_n)$ has density function $c(u_1, u_2, \cdots, u_n)$ which can be decomposed by conditioning into

$$c(u_1, u_2, \cdots, u_n) = c_{n|n-1,\cdots,1}(u_n|u_{n-1}, \cdots, u_1) \cdots c_{2|1}(u_2|u_1)c_1(u_1)$$

where c_1 is the density of U_1. The classical procedure of simulating vector $\mathbf{U}$ is: simulate u_1 based on $c_1(u_1)$, simulate U_2 based on $c_{2|1}(u_2|u_1)$, $\cdots$, simulate U_n based on $c_{n|n-1,\cdots,1}(u_n|u_{n-1}, \cdots, u_1)$. Hence, vector $\mathbf{U}$ is created by calculating $(n-1)$ times the inverses of conditional distribution functions, which is hard and takes a long time to obtain a result, especially if the distribution function of $\mathbf{U}$ is an Archimedean copula. To solve these problems, there are two effective simulation techniques introduced in McNeil et al.[14] and in Brechmann et al.[9].

Stochastic representation of Archimedean copula via a uniform vector on unit simplex and a positive radius

In 2009, McNeil et al [14] showed that there is one-one mapping from the set of n dimensional Archimedean copula generators and the set of n-monotone functions.

Theorem 4.5 (McNeil and Neslehova, 2009)**.** *A real function Φ: $[0,\infty) \to [0,1]$ is the generator of an n dimensional Archimedean copula if and only if it is an $n-$monotone function on $[0,\infty)$ i.e it is differentiable up to the order $(n-2)$ and the derivatives satisfy $(-1)^i\Phi^{(i)}(x) \geq 0, i = 0,1,\cdots,n-2$, further $(-1)^{n-2}\Phi^{(n-2)}$ is non-increasing and convex.*

According to Theorem 4.5, McNeil et al. introduced a stochastic representation of vector $\mathbf{U}$ following an Archimedean copula via a positive radius R and a uniform vector on unit simplex $\mathbb{S}_n = \{w_1,\cdots,w_n \in (0,1)^n : \sum_{i=1}^{n} w_i = 1\}$.

Theorem 4.6 (McNeil and Neslehova, 2009)**.** *If $\boldsymbol{U} = (U_1,\cdots,U_n)$ follows an Archimedean copula with generator Φ and $W = (W_1,\cdots,W_n)$ is an uniform vector on unit simplex $\mathbb{S}_n = \{w_1,\cdots,w_n \in (0,1)^n : \sum_{i=1}^{n} w_i = 1\}$ then there exists a positive random variable R with distribution function satisfies*

$$F_R(x) = 1 - \sum_{j=0}^{n-2}(-1)^j\frac{x^j}{j!}\Phi^{(j)}(x) - (-1)^{n-1}\frac{x^{n-1}}{(n-1)!}\Phi^{(n-1)+}(x)$$

such that

$$(U_1,\cdots,U_n) \stackrel{d}{=} (\Phi(RW_1),\cdots,\Phi(RW_n))$$

As a result of Theorem 4.6 we have

- If vector $\mathbf{X} = (X_1,\cdots,X_n)$ has marginal distribution function $F_1,\cdots,F_n$ and the dependence is an Archimedean copula C of generator Φ, there exists a vector $\mathbf{W}$ uniformly distributed on $\mathbb{S}_n$ and a positive random variable R of distribution function F_R in 4.6 such that

$$(X_1,\cdots,X_n) \stackrel{d}{=} (F_1^{\leftarrow}(\Phi(RW_1)),\cdots,F_n^{\leftarrow}(\Phi(RW_n))).$$

- Similarly, if vector $\mathbf{Y} = (Y_1,\cdots,Y_n)$ with marginal distribution functions $F_1,\cdots,F_n$ and the dependence is an Archimedean survival copula C of generator Φ, then

$$(Y_1,\cdots,Y_n) \stackrel{d}{=} (\bar{F}_1^{\leftarrow}(\Phi(RW_1)),\cdots,\bar{F}_n^{\leftarrow}(\Phi(RW_n))).$$

Archimedean copula simulation based on the Kendall distribution function

Another way to simulate a vector $\mathbf{X}$ and a vector $\mathbf{Y}$ of marginal distribution functions $F_1, \cdots, F_n$ where dependence follows an Archimedean copula and an Archimedean survival copula is the method of Brechmann et al. [9]. Arguing that the classical method does not work due to the problem of calculating the inverse functions of conditional distributions $C_{j|j-1,\cdots,1}$, Brechmann provided an algorithm to simulate Archimedean copula using an intermediate variable Z whose distribution function is known as Kendall distribution function (see Barbe et al. [6]). This method eliminates the problem of numerical calculations of the inverse functions of $C_{j|j-1,\cdots,1}$.

Proposition 4.7 (Brechmann et al., 2013)**.** *If $\boldsymbol{U} = (U_1, \cdots, U_n)$ follows an Archimedean copula C of generator Φ and variable Z is defined as $Z = C(\boldsymbol{U})$, then the density of Z is*

$$f_Z(z) = \frac{(-1)^{n-1}}{(n-1)!} \left[\Phi^{\leftarrow}(z)\right]^{n-1} \left(\Phi^{\leftarrow}\right)^{(1)}(z)\ \Phi^{(n)}\big(\Phi^{\leftarrow}(z)\big)$$

Moreover, conditional distributions $U_j|Z, U_{j-1}, \cdots, U_1$ with $j = 1, \cdots, d$ are

$$F_{U_j|Z,U_{j-1},\cdots,U_1}(u_j|z, u_{j-1}, \cdots, u_1) = \Big(1 - \frac{\Phi^{\leftarrow}(u_j)}{\Phi^{\leftarrow}(z) - \sum_{k=1}^{j-1} \Phi^{\leftarrow}(u_k)}\Big)^{d-j}$$

for $1 > u_j \geq \Phi\big(\Phi^{\leftarrow}(z) - \sum_{k=1}^{j-1} \Phi^{\leftarrow}(u_k)\big)$.

From these results, the inverse function of conditional distribution $F_{U_j|Z,U_{j-1},\cdots,U_1}(u_j|z, u_{j-1}, \cdots, u_1)$ can be calculated by an explicit formula. Indeed, if we have $z, u_{j-1}, \cdots, u_1$ are realisations of $Z, U_{j-1}, \cdots, U_1$ and w is a realisation of a uniform random variable in $(0, 1)$, U_j is simulated as follows

$$\begin{aligned}
& u_j = F^{\leftarrow}_{U_j|Z,U_{j-1},\cdots,U_1}(w|z, u_{j-1}, \cdots, u_1) \\
\rightarrow\ & \Phi^{\leftarrow}(u_j) = (1 - w^{1/(d-j)})\big(\Phi^{\leftarrow}(z) - \sum_{k=1}^{j-1} \Phi^{\leftarrow}(u_k)\big) \\
\rightarrow\ & u_j = \Phi\left((1 - w^{1/(d-j)})\big(\Phi^{\leftarrow}(z) - \sum_{k=1}^{j-1} \Phi^{\leftarrow}(u_k)\big)\right)
\end{aligned}$$

According to this result, Brechmann et al. proposed that this method can be applied to simulate a conditional Archimedean copula $\mathbf{U} = (U_1, \cdots, U_n | U_1 \in [a, b])$ where $[a, b] \in (0, 1)$. To do this, in the first step, we simulate U_1 uniformly distributed in $[a, b]$, then variable Z is simulated via the conditional distribution defined by the following proposition:

Proposition 4.8. *Conditional distribution $F_{Z|U_1}(z|u_1)$ can be calculated by the Archimedean generator and its derivatives*

$$F_{Z|U_1}(z|u1) = \left(\Phi^{\leftarrow}\right)^{(1)}(u_1)\sum_{j=0}^{n-2}\frac{(-1)^j}{j!}\left(\Phi^{\leftarrow}(z)-\Phi^{\leftarrow}(u_1)\right)^j\Phi^{(j+1)}(\Phi^{\leftarrow}(z)) \quad \textit{for } z\in(0,u_1)$$

(see the proof in section 4.5).

Thus, the procedure of simulating conditional Archimedean vector $\mathbf{U} = (U_1, \cdots, U_n | U_1 \in [a, b])$ is: first, simulate uniform variable U_1 satisfying $U_1 \in [a, b]$; second, based on conditional distribution $F_{Z|U_1}$ in 4.8 to simulate Z numerically; finally, based on conditional distribution $F_{U_j|Z,U_{j-1},\cdots,U_1}$ in 4.7 to simulate $U_2, \cdots, U_n$ respectively.

4.3 Our estimators

Before discussing the estimators in details, we define some notations frequently used in this section:

- If $\mathbf{x}$ is an n dimensional vector $\mathbf{x} = (x_1, \cdots, x_n)$, $\mathbf{x}_{-i}$ is the vector $\mathbf{x}$ after removing the i^{th} component x_i.
- We note $M_i(x_1, \cdots, x_n)$ is the i^{th} element of vector $\mathbf{x}$ after arranging the elements of $\mathbf{x}$ in the non-decreasing order. Obviously, $M_1(\mathbf{x}) = \min(\mathbf{x})$ and $M_n(\mathbf{x}) = \max(\mathbf{x})$.
- The notations $z^X(s), S_n^X, M_n^X, \cdots$ refer to $z(s), S_n, M_n, \cdots$ in the case of Archimedean copula. Similarly, $z^Y(s), S_n^Y, M_n^Y, \cdots$ denote $z(s), S_n, M_n, \cdots$ in the case of Archimedean survival copula.
- Probabilities $\mathbb{P}(M_n^X > s)$ and $\mathbb{P}(M_n^Y > s)$ can be calculated by marginal distributions $F_1, \cdots, F_n$ and Archimedean generator Φ. If the dependent is an Archimedean copula C, we have

$$\begin{aligned}\mathbb{P}(M_n^X > s) &= 1 - \mathbb{P}(M_n^X \le s) = 1 - \mathbb{P}(X_1 \le s, \cdots, X_n \le s)\\ &= 1 - \mathbb{P}(F_1(X_1) \le F_1(s), \cdots, F_n(X_n) \le F_n(s))\\ &= 1 - C\left(F_1(s), F_2(s), \cdots, F_n(s)\right).\end{aligned}$$

If the dependence is an Archimedean survival C, then there exits a vector U following an Archimedean copula C such that

$$\begin{aligned}\mathbb{P}(M_n^Y > s) &= 1 - \mathbb{P}(M_n^Y \leq s) = 1 - \mathbb{P}\left(Y_1 \leq s, \cdots, Y_n \leq s\right) \\ &= 1 - \mathbb{P}\left(\bar{F}_1(Y_1) \geq \bar{F}_1(s), \cdots, \bar{F}_n(Y_n) \geq \bar{F}_n(s)\right) \\ &= 1 - \mathbb{P}\left(U_1 \geq \bar{F}_1(s), \cdots, U_n \geq \bar{F}_n(s)\right) \\ &= 1 - \bar{C}\left(\bar{F}_1(s), \bar{F}_2(s), \cdots, \bar{F}_n(s)\right).\end{aligned}$$

The first estimator

Our first estimator is based on the simulation techniques that Brechmann et al. [9] used to simulate conditional vector $\mathbf{U}|U_1 \in [a, b]$ where $\mathbf{U}$ follows an Archimedean copula. After taking probabilities $\mathbb{P}(M_n^X > s)$ and $\mathbb{P}(M_n^Y > s)$ out, we apply the exchangeable property between variables in a sum that Asmussen and Kroese used in their estimator. If the dependence between regularly varying variables is an Archimedean copula,

$$\begin{aligned}z^X(s) &= \mathbb{P}(M_n^X > s) + \mathbb{P}(S_n^X > s, s/n < M_n^X \leq s) \\ &= \mathbb{P}(M_n^X > s) + \sum_{i=1}^{n} \mathbb{P}(S_n^X > s, X_i = M_n^X, s/n < M_n^X \leq s) \\ &= \mathbb{P}(M_n^X > s) + \sum_{i=1}^{n} \mathbb{P}(S_n^X > s, X_i = M_n^X, s/n < X_i \leq s) \\ &= \mathbb{P}(M_n^X > s) + \sum_{i=1}^{n} \mathbb{P}(s/n < X_i \leq s)\ \mathbb{P}(S_n^X > s, X_i = M_n^X | s/n < X_i \leq s).\end{aligned}$$

Consequently, we propose the first estimator for $z^X(s)$

$$Z_{NR1}^X(s) = \mathbb{P}(M_n^X > s) + \sum_{i=1}^{n} \left(\bar{F}_i(s/n) - \bar{F}_i(s)\right)\ \mathbb{I}_{\{S_n^{X_i} > s, X_i^i = M_n^{X_i}\}}$$

where $S_n^{X_i}$, X_i^i and $M_n^{X_i}$ are S_n^X, X_i and M_n^X conditioning on $s/n < X_i \leq s$. The most challenging problem here is the technique of simulating vector $(X_1^i, \cdots, X_n^i) = (X_1, \cdots, X_n | s/n < X_i \leq s)$ where the dependence of $\mathbf{X}$ is an Archimedean copula. Via transformations $U_i \overset{d}{=} F_i(X_i)$, conditional vectors $\mathbf{X}|X_i \in (s/n, s)$ become

$$\begin{aligned}(X_1, \cdots, X_n | s/n < X_i \leq s) &\overset{d}{=} (F_1^{\leftarrow}(U_1), \cdots, F_n^{\leftarrow}(U_n) | s/n < F_i^{\leftarrow}(U_i) \leq s) \\ &\overset{d}{=} (F_1^{\leftarrow}(U_1), \cdots, F_n^{\leftarrow}(U_n) | F_i(s/n) < U_i \leq F_i(s)) \\ &\overset{d}{=} (F_1^{\leftarrow}(U_1^{i+}), \cdots, F_n^{\leftarrow}(U_n^{i+}))\end{aligned}$$

where $(U_1^{i+},\cdots,U_n^{i+}) = (U_1,\cdots,U_n|F_i(s/n) < U_i \leq F_i(s))$. Similarly, if $\mathbf{Y} = (Y_1,\cdots,Y_n)$ follows an Archimedean survival copula C, then

$$\begin{aligned}(Y_1,\cdots,Y_n|s/n < Y_i \leq s) &\overset{d}{=} (\bar{F}_1^{\leftarrow}(U_1),\cdots,\bar{F}_n^{\leftarrow}(U_n)|s/n < \bar{F}_i^{\leftarrow}(U_i) \leq s)\\ &\overset{d}{=} (\bar{F}_1^{\leftarrow}(U_1),\cdots,\bar{F}_n^{\leftarrow}(U_n)|\bar{F}_i(s/n) > U_i \geq \bar{F}_i(s))\\ &\overset{d}{=} (\bar{F}_1^{\leftarrow}(U_1^{i-}),\cdots,\bar{F}_n^{\leftarrow}(U_n^{i-}))\end{aligned}$$

where $(U_1^{i-},\cdots,U_n^{i-}) = (U_1,\cdots,U_n|\bar{F}_i(s/n) > U_i \geq \bar{F}_i(s))$. With conditional vectors $\mathbf{Y}|s/n < Y_i \leq s$, we have the first estimator for $z^Y(s)$:

$$Z_{NR1}^Y(s) = \mathbb{P}(M_n^Y > s) + \sum_{i=1}^{n}\left(\bar{F}_i(s/n) - \bar{F}_i(s)\right)\mathbb{I}_{\{S_n^{Y_i}>s, Y_i^i = M_n^{Y_i}\}}$$

where $S_n^{Y_i}$, Y_i^i and $M_n^{Y_i}$ are S_n^Y, Y_i and M_n^Y conditioning on $s/n < Y_i \leq s$.

Proposition 4.9. *Estimators $Z_{NR1}^X(s)$ and $Z_{NR1}^Y(s)$ have bounded relative errors. (see the proofs in Section 4.5.)*

Algorithm for $Z_{NR1}(s)$

- *Archimedean copula*
  - *For $i = 1,2,\cdots,n$; independently simulate U_i^{i+} uniformly distributed in $(F_i(s/n), F_i(s))$.*
  - *For each U_i^{i+} in the first step, simulate Z based on conditional distribution $F_{Z|U_i}$ and then simulate $(U_1^{i+},\cdots,U_{i-1}^{i+},U_{i+1}^{i+},U_n^{i+})$.*
  - *For each $j = 1,2,\cdots,n,$, calculate $X_j^i = F_j^{\leftarrow}(U_j^{i+})$ and return $\mathbb{I}_{\{S_n^{X_i}>s, X_i^i = M_n^{X_i}\}}$ which takes values 0 or 1.*
  - *Return $Z_{NR1}^X(s) = \mathbb{P}(M_n^X > s) + \sum_{i=1}^n \left(\bar{F}_i(s/n) - \bar{F}_i(s)\right)\mathbb{I}_{\{S_n^{X_i}>s, X_i^i = M_n^{X_i}\}}$.*
- *Archimedean survival copula*
  - *For $i = 1,2,\cdots,n$; independently simulate U_i^{i-} uniformly distributed in $(\bar{F}_i(s/n), \bar{F}_i(s))$.*
  - *For each U_i^{i-} in the first step, simulate Z based on conditional distribution $F_{Z|U_i}$ and then simulate $(U_1^{i-},\cdots,U_{i-1}^{i-},U_{i+1}^{i-},U_n^{i-})$.*
  - *For each $j = 1,2,\cdots,n,$, calculate $Y_j^i = F_j^{\leftarrow}(U_j^{i-})$ and return $\mathbb{I}_{\{S_n^{Y_i}>s, Y_i^i = M_n^{Y_i}\}}$ which takes values 0 or 1.*
  - *Return $Z_{NR1}^Y(s) = \mathbb{P}(M_n^Y > s) + \sum_{i=1}^n \left(\bar{F}_i(s/n) - \bar{F}_i(s)\right)\mathbb{I}_{\{S_n^{Y_i}>s, Y_i^i = M_n^{Y_i}\}}$.*

It is remarkable that the technique of estimating $Z_{NR1}(s)$ can be applied for the sums of regularly varying variables with any dependence structure and the estimators have the bounded relative error. The key solution is how to simulate conditional random vector $\mathbf{X}|s/n < X_i \leq s$. Unfortunately, numerical performance of $Z_{NR1}(s)$ presented in Section 4.4 is not as good as the other estimators.

The second estimator

The construction of the second estimator is based on the stochastic representation of Archimedean copula by McNeil et al [14]. As stated in Section 4.2, if the dependence of $\mathbf{X}$ is an Archimedean copula, we have

$$(X_1, \cdots, X_n) \stackrel{d}{=} (F_1^{\leftarrow}(\Phi(RW_1)), \cdots, F_n^{\leftarrow}(\Phi(RW_n)))$$

where the distribution function of R is as in Theorem 4.6 and $\mathbf{W}$ is a vector uniformly distributed on $\mathbb{S}_n$. Probability $z^X(s)$ can be written as follows

$$\begin{aligned} z^X(s) &= P(M_n^X > s) + P(S_n^X > s, M_n^X \leq s) \\ &= P(M_n^X > s) + P\left(\sum_{i=1}^n F_i^{\leftarrow}(\Phi(RW_i)) > s, \max_{i=1,\cdots,n}\{F_i^{\leftarrow}(\Phi(RW_i))\} \leq s\right) \\ &= P(M_n^X > s) + P\left(\sum_{i=1}^n F_i^{\leftarrow}(\Phi(RW_i)) > s, R \geq \max_{i=1,\cdots,n}\left\{\frac{\Phi^{\leftarrow}(F_i(s))}{W_i}\right\}\right). \end{aligned}$$

If the dependence between regularly varying marginals is an Archimedean survival copula, where the stochastic representation of $\mathbf{Y}$ can be written according to McNeil et al [14]

$$(Y_1, \cdots, Y_n) \stackrel{d}{=} \left(\bar{F}_1^{\leftarrow}(\Phi(RW_1)), \cdots, \bar{F}_n^{\leftarrow}(\Phi(RW_n))\right),$$

we have probability $z^Y(s)$

$$\begin{aligned} z^Y(s) &= P(M_n^Y > s) + P(S_n^Y > s, M_n^Y \leq s) \\ &= P(M_n^Y > s) + P\left(\sum_{i=1}^n \bar{F}_i^{\leftarrow}(\Phi(RW_i)) > s, \max_{i=1,\cdots,n}\left\{\bar{F}_i^{\leftarrow}(\Phi(RW_i))\right\} \leq s\right) \\ &= P(M_n^Y > s) + P\left(\sum_{i=1}^n \bar{F}_i^{\leftarrow}(\Phi(RW_i)) > s, R \leq \min_{i=1,\cdots,n}\left\{\frac{\Phi^{\leftarrow}(\bar{F}_i(s))}{W_i}\right\}\right). \end{aligned}$$

The second estimators for $z^X(s)$ and $z^Y(S)$ are obtained conditioning on $\mathbf{W}$:

- Under Archimedean copula,

$$Z^X_{NR2}(s) = P(M^X_n > s) + F_R(U^X(\mathbf{W}, s)) - F_R(L^X(\mathbf{W}, s))$$

where

$$U^X(\mathbf{W}, s) = \sup\{r \in \mathbb{R}^+ : \sum_{i=1}^{n} F_i^{\leftarrow}(\Phi(rW_i)) \leq s\}$$

$$L^X(\mathbf{W}, s) = \max_{i=1,\cdots,n} \left\{ \frac{\Phi^{\leftarrow}(F_i(s))}{W_i} \right\}.$$

- Under Archimedean survival copula, we have

$$Z^Y_{NR2}(s) = P(M^Y_n > s) + F_R(U^Y(\mathbf{W}, s)) - F_R(L^Y(\mathbf{W}, s))$$

where

$$U^Y(\mathbf{W}, s) = \min_{i=1,\cdots,n} \left\{ \frac{\Phi^{\leftarrow}(\bar{F}_i(s))}{W_i} \right\}$$

$$L^Y(\mathbf{W}, s) = \inf\{r \in \mathbb{R}^+ : \sum_{i=1}^{n} \bar{F}_i^{\leftarrow}(\Phi(rW_i)) \geq s\}. \tag{4.1}$$

Note that if the marginal distributions are continuous and strictly increasing, then $U^X(\mathbf{W}, s)$ and $L^Y(\mathbf{W}, s)$ are unique roots of equations $\sum_{i=1}^{n} F_i^{\leftarrow}(\Phi(xW_i)) = s$ and $\sum_{i=1}^{n} \bar{F}_i^{\leftarrow}(\Phi(xW_i)) = s$ respectively. The quality of estimators $Z^Y_{NR2}(s)$ will be verified with the assumption that the Archimedean generator is regularly varying.

Proposition 4.10. *If $(Y_1, \cdots, Y_n)$ follows an Archimedean survival copula C of generator Φ satisfies: $\Phi^{(n-2)}$ is differentiable and $\Phi \in \mathcal{RV}_\infty(-\beta); \beta > 0$, then $Z^Y_{NR2}(s)$ has a bounded relative error. (see the proof in Section 4.5).*

Algorithm for $Z_{NR2}(s)$

- *Archimedean copula*
  - *Let $(E_1, E_2, \cdots, E_n)$ be n i.i.d exponential r.v of parameter 1, calculate $W_i = E_i / \sum_{j=1}^{n} E_j$.*
  - *$U^X(\boldsymbol{W}, s)$ is calculated numerically from Equation 4.1 and $L^X(\boldsymbol{W}, s)$ is determined from Equation 4.1*

  - *Calculate derivatives* $\Phi^{(j)}(x)$ *for* $j = 1, \cdots, n-1$ *and then distribution function* $F_R(x)$
  - *Return* $Z^X_{NR2}(s) = P(M^X_n > s) + F_R(U^X(\boldsymbol{W}, s)) - F_R(L^X(\boldsymbol{W}, s))$

- *Archimedean survival copula*
  - *Let* $(E_1, E_2 \cdots, E_n)$ *be* n *i.i.d exponential r.v of parameter 1, calculate* $W_i = E_i / \sum_{j=1}^{n} E_j$.
  - $U^Y(\boldsymbol{W}, s)$ *is determined from Equation 4.1 and then calculate* $L^Y(\boldsymbol{W}, s)$ *numerically from 4.1*
  - *Calculate derivatives* $\Phi^{(j)}(x)$ *for* $j = 1, \cdots, n-1$ *and then distribution function* $F_R(x)$
  - *Return* $Z^Y_{NR2}(s) = P(M^Y_n > s) + F_R(U^Y(\boldsymbol{W}, s)) - F_R(L^Y(\boldsymbol{W}, s))$

The third estimator

This section presents another estimator for $z(s)$ which has better numerical performances than $Z_{NR1}(s)$ and $Z_{NR2}(s)$. We separate probability $z(s)$ into

$$\mathbb{P}(S_n > s) = \mathbb{P}(M_n > s) + z_1(s) + z_2(s)$$

where $\mathbb{P}(S_n > s)$ is deterministic, $z_1(s) = \mathbb{P}(S_n > s, M_{n-1} \leq \lambda s, M_n \leq s)$, $z_2(s) = \mathbb{P}(S_n > s, M_{n-1} > \lambda s, M_n \leq s)$ and λ is a positive less than $1/n$. In $z_1(s)$, inequation $M_{n-1} \leq \lambda s$ implies that there is only one variable taking a large value. Consequently, we estimate $z_1(s)$ conditionally on $\mathbf{X}_{-i}$. In $z_2(s)$, there are at least two variables taking large values, so it is coherent if we estimate $z_2(s)$ conditionally on uniform vector $\mathbf{W}$ on unit simplex $\mathbb{S}^n$.

Estimating $z_2(s)$ conditionally on a uniform vector on $\mathbb{S}^n$

If the dependence structure is an Archimedean copula, according to McNeil et al. [14], the stochastic representation of $(X_1, \cdots, X_n)$ is $(F_1^{\leftarrow}(\Phi(RW_1)), \cdots, F_n^{\leftarrow}(\Phi(RW_n)))$. Hence, probability $z_2^X(s) = \mathbb{P}(S^X_n > s, M^X_{n-1} > \lambda s, M^X_n \leq s)$ becomes

$$\begin{aligned} z_2^X(s) &= \mathbb{P}(\sum_{i=1}^{n} F_i^{\leftarrow}\big(\Phi(W_i R)\big) > s, M_{n-1}\{F_i^{\leftarrow}\big(\Phi(W_i R)\big)\} > \lambda s, M_n\{F_i^{\leftarrow}\big(\Phi(W_i R)\big)\} \leq s) \\ &= \mathbb{P}(R < U^X(\mathbf{W}, s), R < M_{n-1}\{\frac{\Phi^{\leftarrow}\big(F_i(\lambda s)\big)}{W_i}\}, R \geq L^X(\mathbf{W}, s)). \end{aligned}$$

Similarly, under Archimedean survival copula,

$$\begin{aligned} z_2^Y(s) &= \mathbb{P}(\sum_{i=1}^{n} \bar{F}_i^{\leftarrow}\big(\Phi(W_i R)\big) > s, M_{n-1}\{\bar{F}_i^{\leftarrow}\big(\Phi(W_i R)\big)\} \geq \lambda s, M_n\{\bar{F}_i^{\leftarrow}\big(\Phi(W_i R)\big)\} \leq s) \\ &= \mathbb{P}(R > L^Y(\mathbf{W}, s), R \geq M_2\{\frac{\Phi^{\leftarrow}(\bar{F}_i(\lambda s))}{W_i}\}, R \leq U^Y(\mathbf{W}, s)). \end{aligned}$$

Conditioning on $\mathbf{W}$, we have estimator $Z^X_{NR3,2}(s)$ for $z_2^X(s)$

$$\begin{aligned} Z^X_{NR3,2}(s) &= F_R\big(U^X(\mathbf{W}, s) \wedge M_{n-1}\{\frac{\Phi^{\leftarrow}\big(F_i(\lambda s)\big)}{W_i}\}\big) - F_R\big(L^X(\mathbf{W}, s)\big) \\ &= F_R\big(U^X_\lambda(\mathbf{W}, s)\big) - F_R\big(L^X(\mathbf{W}, s)\big) \end{aligned}$$

with $U^X(\mathbf{W}, s)$, $L^X(\mathbf{W}, s)$ are from 4.1, 4.1 respectively, and

$$U^X_\lambda(\mathbf{W}, s) = U^X(\mathbf{W}, s) \wedge M_{n-1}\{\frac{\Phi^{\leftarrow}\big(F_i(\lambda s)\big)}{W_i}\}. \tag{4.2}$$

Under Archimedean survival copula,

$$\begin{aligned} Z^Y_{NR3,2}(s) &= F_R\big(U^Y(\mathbf{W}, s)\big) - F_R\big(L^Y(\mathbf{W}, s) \vee M_2\{\frac{\Phi^{\leftarrow}(\bar{F}_i(\lambda s))}{W_i}\}\big) \\ &= F_R\big(U^Y(\mathbf{W}, s)\big) - F_R\big(L^Y_\lambda(\mathbf{W}, s)\big) \end{aligned}$$

with $U^Y(\mathbf{W}, s)$, $L^Y(\mathbf{W}, s)$ are from 4.1, 4.1 respectively, and

$$L^Y_\lambda(\mathbf{W}, s) = L^Y(\mathbf{W}, s) \vee M_2\{\frac{\Phi^{\leftarrow}(\bar{F}_i(\lambda s))}{W_i}\}. \tag{4.3}$$

Proposition 4.11. *There exists a constant c such that the variance of $Z^Y_{NR3,2}(s)$ is asymptotically bounded by $c \times z(s)$. (see the proof in Section 4.5).*

Asmussen and Kroese's method to estimate $z_1(s)$.

The idea of Asmussen and Kroese's method is to use M_n^X as variable controlling S_n^X. We develop probability $z_1(s)$ as follows

$$\begin{aligned} \mathbb{P}(S_n^X > s, M_{n-1}^X < \lambda s, M_n^X < s) &= \sum_{i=1}^{n} \mathbb{P}(S_n^X > s, M_{n-1}^X < \lambda s, M_n^X < s, X_i = M_n^X) \\ &= \sum_{i=1}^{n} \mathbb{P}(S_n^X > s, max\{\mathbf{X}_{-i}\} < \lambda s, X_i < s, X_i = M_n^X) \end{aligned}$$

For each $i = 1, 2, \cdots, n$, we propose the use of estimator $Z^{[i]X}_{NR3,1}(s)$ for $\mathbb{P}(S^X_n > s, max\{\mathbf{X}_{-i}\} < \lambda s, X_i < s, X_i = M^X_n)$ by conditioning on $\mathbf{X}_{-i} = \mathbf{x}_{-i}$:

$$Z^{[i]X}_{NR3,1}(s) = \mathbb{I}_{\{max\{\mathbf{x}_{-i}\}<\lambda s\}}\ \mathbb{P}\left(s > X^*_i > s - \sum_{j=1,j\neq i}^{n} x_j\right)$$

where $X^*_i = X_i|\mathbf{X}_{-i} = \mathbf{x}_{-i}$. Note that if $\max\{\mathbf{x}_{-i}\} < \lambda s$ then

$$s - \sum_{j=1,j\neq i}^{n} x_j > (1-(n-1)\lambda)s > s/n > \lambda s > \max\{\mathbf{x}_{-i}\}$$

which implies condition $X_i > \max\{\mathbf{X}_{-i}\}$ is definitely correct. Estimator $Z^X_{NR3,1}(s)$ for $z^X_1(s)$ is then defined by

$$Z^X_{NR3,1}(s) = \sum_{i=1}^{n} Z^{[i]X}_{NR3,1}(s).$$

In the case of marginal distributions are identical, we have $Z^X_{NR3,1}(s) = n\mathbb{I}_{\{\max\{\mathbf{x}_{-1}\}<\lambda s\}}\ \mathbb{P}(s > X^*_1 > s - sum(\mathbf{x}_{-i}))$ as original Asmussen and Kroese's estimator. Under the assumption that the dependence follows an Archimedean survival copula, we employ the same method.

To perform the calculations, we need conditional distributions $X^*_i = X_i|\mathbf{X}_{-i} = \mathbf{x}_{-i}$ and $Y^*_i = Y_i|\mathbf{Y}_{-i} = \mathbf{y}_{-i}$. The distribution of X^*_i can be obtained directly

$$F_{X^*_i}(x_i) = \mathbb{P}(X^*_i \leq x_i) = \frac{\Phi^{(n-1)}\big(\sum_{j=1}^{n}\Phi^{-1}(F_j(x_j))\big)}{\Phi^{(n-1)}\big(\sum_{j=1,j\neq i}^{n}\Phi^{-1}(F_j(x_j))\big)}$$

However, in the case of Archimedean survival copula, the expression for distribution of Y^*_i is more complex.

Proposition 4.12. *If random variables $Y_1, \cdots, Y_n$ of distribution functions $F_1, \cdots, F_n$ with the dependence follows an Archimedean survival copula C, then the distribution function of $Y^*_i = Y_i|\,\boldsymbol{Y}_{-i} = \boldsymbol{y}_{-i}$ is*

$$F_{Y^*_i}(y_i) = \mathbb{P}(Y^*_i \leq y_i) = 1 - \frac{\Phi^{(n-1)}\big(\sum_{j=1}^{n}\Phi^{-1}(\bar{F}_j(y_j))\big)}{\Phi^{(n-1)}\big(\sum_{j=1,j\neq i}^{n}\Phi^{-1}(\bar{F}_j(y_j))\big)}$$

(see the proof in Section 4.5)

Unfortunately, the relative error of $Z_{NR3,1}(s)$ is not bounded if there is no assumption for Archimedean generator. Consequently, the relative error of $Z_{NR3}(s)$ is not bounded either. However, numerical performances of this estimator are better than $Z_{NR2}(s)$ in some situations

when parameter λ takes appropriate values. Moreover, in almost all cases, $Z_{NR3}(s)$ is better than $Z_{NR1}(s)$ which is proved to have a bounded relative error.

Algorithm for $Z_{NR3}(s)$

- *Archimedean copula*
  - *Let $(E_1, E_2 \cdots, E_n)$ be n i.i.d exponential r.v of parameter 1, calculate $W_i = E_i / \sum_{j=1}^{n} E_j$.*
  - *Calculate $U_\lambda^X(\boldsymbol{W}, s)$ from 4.2 and $L^X(\boldsymbol{W}, s)$ from 4.1*
  - *For $i = 1, 2, \cdots, n$; simulate vector $\boldsymbol{U}_{-i}$ following $(n-1)$ dimensional Archimedean copula of generator Φ and then calculate $X_j = F_j^{\leftarrow}(U_j)$ for $j \neq i$. After that, calculate the value of $Z_{NR3,1}^{[i]X}(s) = \mathbb{I}_{\{max\{\boldsymbol{x}_{-i}\} < \lambda s\}} \left(F_{X_i^*}(s) - F_{X_i^*}(s - sum(\boldsymbol{x}_{-i}))\right)$*
  - *Return $Z_{NR3}^X(s) = P(M_n^X > s) + F_R(U_\lambda^X(\boldsymbol{W}, s)) - F_R(L^X(\boldsymbol{W}, s)) + \sum_{i=1}^{n} Z_{NR3,1}^{[i]X}(s)$*
- *Archimedean survival copula*
  - *Let $(E_1, E_2 \cdots, E_n)$ be n i.i.d exponential r.v of parameter 1, calculate $W_i = E_i / \sum_{j=1}^{n} E_j$.*
  - *Calculate $U^Y(\boldsymbol{W}, s)$ from 4.1 and $L_\lambda^Y(\boldsymbol{W}, s)$ from 4.3*
  - *For $i = 1, 2, \cdots, n$; simulate vector $\boldsymbol{U}_{-i}$ following $(n-1)$ dimensional Archimedean copula of generator Φ and then calculate $Y_j = \bar{F}_j^{\leftarrow}(U_j)$ for $j \neq i$. After that, calculate the value of $Z_{NR3,1}^{[i]Y}(s) = \mathbb{I}_{\{max\{\boldsymbol{y}_{-i}\} < \lambda s\}} \left(F_{Y_i^*}(s) - F_{Y_i^*}(s - sum(\boldsymbol{y}_{-i}))\right)$*
  - *Return $Z_{NR3}^Y(s) = P(M_n^Y > s) + F_R(U^Y(\boldsymbol{W}, s)) - F_R(L_\lambda^Y(\boldsymbol{W}, s)) + \sum_{i=1}^{n} Z_{NR3,1}^{[i]Y}(s)$*

The fourth estimator

In this section, with the same idea of separating probability $\mathbb{P}(S_n > s, M_n \leq s)$ into two parts, we introduce the 4^{th} estimator for $z(s)$, called $Z_{NR4}(s)$, which has bounded relative error under Archimedean survival copula without any assumption of Φ. First, with κ is chosen in $(1/n, 1)$, we decompose probability $z(s)$ into

$$\begin{aligned} z(s) &= \mathbb{P}(M_n > s) + \mathbb{P}(S_n > s, M_n \leq s) \\ &= \mathbb{P}(M_n > s) + \mathbb{P}(S_n > s, \kappa s < M_n \leq s) + \mathbb{P}(S_n > s, M_n \leq \kappa s). \end{aligned}$$

Probability $\mathbb{P}(S_n > s, \kappa s < M_n \leq s)$ will be estimated by the same method of estimating $Z_{NR1}(s)$ while probability $\mathbb{P}(S_n > s, M_n \leq \kappa s)$ will be estimated conditionally on $\mathbf{W} \in \mathbb{S}_n$. If

the dependence structure is an Archimedean copula, we have

$$\begin{aligned} z^X(s) &= \mathbb{P}(M_n^X > s) + \sum_{i=1}^{n} \left(\bar{F}_i(\kappa s) - \bar{F}_i(s)\right) \mathbb{P}(S_n^X > s, X_i = M_n^X | \kappa s < X_i \leq s) \\ &+ \mathbb{P}\left(\sum_{i=1}^{n} F_i^{\leftarrow}(\Phi(RW_i)) > s, R \geq \max_{i=1,\cdots,n}\left\{\frac{\Phi^{\leftarrow}(F_i(\kappa s))}{W_i}\right\}\right). \end{aligned}$$

This formula gives the 4^{th} estimator $Z_{NR4}^X(s)$:

$$Z_{NR4}^X(s) = \mathbb{P}(M_n^X > s) + \sum_{i=1}^{n} \left(\bar{F}_i(\kappa s) - \bar{F}_i(s)\right)\mathbb{I}_{\{S_n^{X\kappa i} > s, X_i^{\kappa i} = M_n^{X\kappa i}\}} + F_R(U^X(\mathbf{W}, s)) - F_R(L_\kappa^X(\mathbf{W}, s))$$

with $X_j^{\kappa i} = X_j | \kappa s < X_i \leq s$; $U^X(\mathbf{W}, s)$ is defined in 4.1 and

$$L_\kappa^X(\mathbf{W}, s) = \max_{i=1,\cdots,n}\left\{\frac{\Phi^{\leftarrow}(F_i(\kappa s))}{W_i}\right\}. \tag{4.4}$$

Similarly, if the dependence is an Archimedean survival copula

$$\begin{aligned} z^Y(s) &= \mathbb{P}(M_n^Y > s) + \sum_{i=1}^{n} \left(\bar{F}_i(\kappa s) - \bar{F}_i(s)\right) \mathbb{P}(S_n^Y > s, Y_i = M_n^Y | \kappa s < Y_i \leq s) \\ &+ P\left(\sum_{i=1}^{n} \bar{F}_i^{\leftarrow}(\Phi(RW_i)) > s, R \leq \min_{i=1,\cdots,n}\left\{\frac{\Phi^{\leftarrow}(\bar{F}_i(\kappa s))}{W_i}\right\}\right) \end{aligned}$$

which gives the 4^{th} estimator $Z_{NR4}^Y(s)$

$$Z_{NR4}^Y(s) = \mathbb{P}(M_n^Y > s) + \sum_{i=1}^{n} \left(\bar{F}_i(\kappa s) - \bar{F}_i(s)\right)\mathbb{I}_{\{S_n^{Y\kappa i} > s, Y_i^{\kappa i} = M_n^{Y\kappa i}\}} + F_R(U_\kappa^Y(\mathbf{W}, s)) - F_R(L^Y(\mathbf{W}, s))$$

with $Y_j^{\kappa i} = Y_j | \kappa s < Y_i \leq s$; $L^Y(\mathbf{W}, s)$ is defined in 4.1 and

$$U_\kappa^Y(\mathbf{W}, s) = \min_{i=1,\cdots,n}\left\{\frac{\Phi^{\leftarrow}(\bar{F}_i(\kappa s))}{W_i}\right\}. \tag{4.5}$$

Proposition 4.13. *$Z_{NR4}^Y(s)$ is an estimator with bounded relative error. (see the proof in Section 4.5)*

Algorithm for $Z_{NR4}(s)$

- *Archimedean copula*

- *For each* $i = 1, 2, \cdots, n$, *simulate vector* $(X_1^{\kappa i}, \cdots, X_n^{\kappa i}) = (X_1, \cdots, X_n | \kappa s < X_i \leq s)$, *then calculate* $Z^X_{NR4,1}(s) = \sum_{i=1}^{n} \left(\bar{F}_i(\kappa s) - \bar{F}_i(s)\right) \mathbb{I}_{\{S_n^{X\kappa i} > s, X_i^{\kappa i} = M_n^{X\kappa i}\}}$.
- *Calculate* $U^X(\boldsymbol{W}, s)$ *from 4.1 and* $L^X_\kappa(\boldsymbol{W}, s)$ *from 4.4.*
- *Return* $Z^X_{NR4}(s) = P(M_n^X > s) + Z^X_{NR4,1}(s) + \bar{F}_R(U^X(\boldsymbol{W}, s)) - \bar{F}_R(L^X_\kappa(\boldsymbol{W}, s))$.

- *Archimedean survival copula*
  - *For each* $i = 1, 2, \cdots, n$, *simulate vector* $(Y_1^{\kappa i}, \cdots, Y_n^{\kappa i}) = (Y_1, \cdots, Y_n | \kappa s < Y_i \leq s)$, *then calculate* $Z^Y_{NR4,1}(s) = \sum_{i=1}^{n} \left(\bar{F}_i(\kappa s) - \bar{F}_i(s)\right) \mathbb{I}_{\{S_n^{Y\kappa i} > s, Y_i^{\kappa i} = M_n^{Y\kappa i}\}}$.
  - *Calculate* $U^Y_\kappa(\boldsymbol{W}, s)$ *from 4.5 and* $L^Y(\boldsymbol{W}, s)$ *from 4.1.*
  - *Return* $Z^Y_{NR4}(s) = P(M_n^Y > s) + Z^Y_{NR4,1}(s) + \bar{F}_R(U^Y_\kappa(\boldsymbol{W}, s)) - \bar{F}_R(L^Y(\boldsymbol{W}, s))$.

4.4 Numerical studies

The numerical performances of the 4 estimators are discussed in this section. The results contain the calculations of the probability that the sum of **5** *Lomax* random variables of the same tail index - 2.5 and the same parameter $\beta = 1$ is larger than s. The dependence is assumed to be a Clayton copula, a Gumbel copula, a Clayton survival copula and a Gumbel survival copula. Even in some cases, the quality of the estimators is not verified; we still do the simulation because their numerical performances are favorable. For estimators $Z_{NR3}(s)$ and $Z_{NR4}(s)$, the choices of λ and κ are sensitive. In fact, we choose the values that minimize the numerical standard deviations of the estimators.

Clayton copula

The generator of Clayton copula of parameter $\theta \in (0, \infty)$ and the inverse function:

$$\Phi(t) = \left(1 + \frac{t}{\theta}\right)^{-\theta} \; ; \; \Phi^{\leftarrow}(t) = \theta\left(t^{-1/\theta} - 1\right) \; ; \; \left(\Phi^{\leftarrow}\right)^{(1)}(t) = -t^{-1/\theta - 1}.$$

The formula for n-dimensional Clayton copula

$$C(u_1, \cdots, u_n) = \left(u_1^{-1/\theta} + \cdots + u_n^{-1/\theta} - (n-1)\right)^{-\theta}$$

The derivatives of the generator are calculated as follows

$$\Phi^{(k)}(t) = \left(1+\frac{1}{\theta}\right)\left(1+\frac{2}{\theta}\right)\cdots\left(1+\frac{k-1}{\theta}\right)\left(1+\frac{t}{\theta}\right)^{-\theta-k+1}$$

The Clayton copula has the generator satisfying the Proposition 4.10: $\Phi \in \mathcal{RV}_\infty(-\theta)$ with $\theta > 0$, i.e estimator $Z^Y_{NR2}(s)$ has bounded relative error under Clayton survival copula.

Gumbel copula

The generator of Gumbel copula of the parameter $b \in (0,1)$ and the inverse function:

$$\Phi(t) = \exp(-x^b)\ ;\ \Phi^{\leftarrow}(t) = (-\log(t))^{1/b}\ ;\ \Phi^{\leftarrow}(t) = \frac{1}{bt}(-\log(t))^{1/b-1}\,.$$

The formula of n-dimensional Clayton copula:

$$C(u_1,\cdots,u_n) = \exp\left(-\left[(-\log(u_1))^{1/b}+\cdots+(-\log(u_n))^{1/b}\right]^b\right).$$

It is hard to derive a closed form for the k^{th} derivative of Gumbel copula generator. In this section, with $n = 5$, we have to do the calculations until the 4^{th} derivative:

$$\begin{aligned}
\Phi^{(1)}(t) &= \exp(-x^b)\left(-bt^{b-1}\right)\\
\Phi^{(2)}(t) &= \exp(-t^b)\left(-b(b-1)t^{b-2}+b^2t^{2b-2}\right)\\
\Phi^{(3)}(t) &= \exp(-t^b)\left(-b(b-1)(b-2)t^{b-3}+3b^2(b-1)t^{2b-3}-b^3t^{3b-3}\right)\\
\Phi^{(4)}(t) &= \exp(-t^b)\times\\
&\quad\left(-b(b-1)(b-2)(b-3)t^{b-4}+b^2(b-1)(7b-11)t^{2b-4}-6b^3(b-1)t^{3b-4}+b^4t^{4b-4}\right).
\end{aligned}$$

There are three levels of copulas dependence in this section using Kendall's τ: The weak level of dependence is when $\tau = 0.1$, the normal level of dependence is when $\tau = 0.5$ and the strong level of dependence is when $\tau = 0.9$. Parameter θ of Clayton copula and parameter b of Gumbel copula are determined via τ

$$\theta = \frac{1-\tau}{2\tau} \quad and \quad b = 1-\tau.$$

According to the numerical results, it is remarkable that Z_{NR1} has bounded relative error. For example, under the assumption that the dependence is a Clayton survival copula with Kendall's τ is equal to 0.5, when s increases from 20 (Table 4.3) to 200 (Table 4.6), the value of

$z(s)$ decreases from 0.01639236 to 8.67011E-05, but the relative error of Z_{NR1} does not change: 2.034 compared to 2.036.

Although $Z_{NR1}(s)$ is proved to have a bounded relative error under any dependence structure, the numerical performances of this estimator is not better than $Z_{NR2}(s)$. Note that Z_{NR2} has bounded relative error only when the dependence structure is an Archimedean survival copula of generator $\Phi \in \mathcal{RV}_{\infty}(\beta)$, that is the case of Clayton survival copula in this section. However, except the case of Clayton copula, Z_{NR2} presents acceptable results in most cases. For example, in Table 4.3, under Gumbel survival copula, ratio $\frac{e(Z_{NR1})}{e(Z_{NR2})}$ equals to $\frac{2.378}{0.065} \approx 37$; or in Table 4.7, this ratio under Gumbel copula is approximated to $\frac{1.973}{0.134} \approx 15$.

The construction of Z_{NR3} is more complex than that of Z_{NR2}; however, the 3^{rd} estimator has no numerical improvement compared to the 2^{nd} one except for the case of Clayton copula. Indeed, in Table 4.5, the relative error of Z_{NR3} is 0.480 while the relative error of Z_{NR2} is 3.396 or in Table 4.6, the relative error of Z_{NR3} is 0.182 while the relative error of Z_{NR2} is 2.882. Under the other dependence structures, the relative errors of Z_{NR3} and Z_{NR2} are almost the same.

The 4^{th} estimator has bounded relative error under Archimedean survival copula and it presents favorable numerical results even when the dependence structure is an Archimedean copula. For example, Z_{NR4} has the smallest relative error under Clayton copula in all tables. Under Gumbel copula, except Table 4.4 where $s = 20$ and Kendall's $\tau = 0.9$ or Table 4.7 where $s = 200$ and Kendall's $\tau = 0.9$, Z_{NR4} also has the smallest relative error. Under Archimedean survival copulas, there is not much difference between the relative error of Z_{NR2}, Z_{NR3} and Z_{NR4}.

4.5 Intermediary proofs

Proof of Proposition 4.8

From the conditional distribution $F_{U_j|Z,U_{j-1},\cdots,U_1}(u_j|z,u_{j-1},\cdots,u_1)$ with j = 1 we have the conditional distribution $F_{U_1|Z}$

$$F_{U_1|Z}(u_1|z) = \big(1 - \frac{\Phi^{\leftarrow}(u_1)}{\Phi^{\leftarrow}(z)}\big)^{d-1}$$

for $z < u_1 < 1$. Because the marginal density of U_1 is 1 on $(0,1)$, with the density of Z in Proposition 4.7, we have the conditional density of $Z|U_1$

$$f_{Z|U_1}(z|u1) = \frac{(\Phi^{\leftarrow})^{(1)}(u_1)}{(n-2)!}\big(\Phi^{\leftarrow}(u_1) - \Phi^{\leftarrow}(z)\big)^{n-2}(\Phi^{\leftarrow})^{(1)}(z)\Phi^{(n)}\big(\Phi^{\leftarrow}(z)\big)$$

for $0 < z < u_1$. The distribution function of Z is calculated as

$$\begin{aligned}
F_{Z|U_1}(z|u1) &= \big(\Phi^{\leftarrow}\big)^{(1)}(u_1)\frac{(-1)^{n-2}}{(n-2)!}\int_0^z \big(\Phi^{\leftarrow}(v) - \Phi^{\leftarrow}(u_1)\big)^{n-2}\big(\Phi^{\leftarrow}\big)^{(1)}(v)\Phi^{(n)}\big(\Phi^{\leftarrow}(v)\big)dv \\
&= -\big(\Phi^{\leftarrow}\big)^{(1)}(u_1)\frac{(-1)^{n-2}}{(n-2)!}\int_{\Phi^{\leftarrow}(z)}^{\infty} \big(v - \Phi^{\leftarrow}(u_1)\big)^{n-2}\Phi^{(n)}(v)dv \\
&= -\big(\Phi^{\leftarrow}\big)^{(1)}(u_1)\frac{(-1)^{n-2}}{(n-2)!}\int_{\Phi^{\leftarrow}(z)}^{\infty} \big(v - \Phi^{\leftarrow}(u_1)\big)^{n-2}d\big(\Phi^{(n-1)}(v)\big) \\
&= -\big(\Phi^{\leftarrow}\big)^{(1)}(u_1)\frac{(-1)^{n-2}}{(n-2)!}\big[\big(v - \Phi^{\leftarrow}(u_1)\big)^{n-2}\Phi^{(n-1)}(v)|_{\Phi^{\leftarrow}(z)}^{\infty} \\
&- \int_{\Phi^{\leftarrow}(z)}^{\infty} (n-2)\big(v - \Phi^{-1}(u_1)\big)^{n-3}\Phi^{(n-1)}(v)dv\big]
\end{aligned}$$

Note that $\lim_{v\to\infty}\big(v - \Phi^{\leftarrow}(u_1)\big)^j\Phi^{(j)}(v) = 0$ for all $j = 1,\cdots,n-2$. The distribution of Z conditioning on U_1 is then

$$\begin{aligned}
F_{Z|U_1}(z|u1) &= \big(\Phi^{\leftarrow}\big)^{(1)}(u_1)\big[\frac{(-1)^{n-2}}{(n-2)!}\big(\Phi^{\leftarrow}(z) - \Phi^{\leftarrow}(u_1)\big)^{n-2}\Phi^{(n-1)}(\Phi^{\leftarrow}(z)) \\
&- \frac{(-1)^{n-3}}{(n-3)!}\int_{\Phi^{\leftarrow}(z)}^{\infty} \big(v - \Phi^{\leftarrow}(u_1)\big)^{n-3}\Phi^{(n-1)}(v)dv\big] \\
&= \cdots\cdots \\
&= \big(\Phi^{\leftarrow}\big)^{(1)}(u_1)\sum_{j=0}^{n-2}\frac{(-1)^j}{j!}\big(\Phi^{\leftarrow}(z) - \Phi^{\leftarrow}(u_1)\big)^j\Phi^{(j+1)}(\Phi^{\leftarrow}(z))
\end{aligned}$$

for $z \in (0, u_1)$.

Proof of Proposition 4.9

We can verify this property easily: the variance of $Z^X_{NR1}(s)$ is bounded by:

$$\begin{aligned}
\mathbb{V}ar(Z^X_{NR1}(s)) &= var\left(\sum_{i=1}^{n}\left(\bar{F}_i(s/n) - \bar{F}_i(s)\right)\mathbb{I}_{\{S_n^{X_i}>s, X_i^i = M_n^{X_i}\}}\right) \\
&= \sum_{i=1}^{n}\left(\bar{F}_i(s/n) - \bar{F}_i(s)\right)^2 var\left(\mathbb{I}_{\{S_n^{X_i}>s, X_i^i = M_n^{X_i}\}}\right) \\
&\leq \sum_{i=1}^{n}[\bar{F}_i(s/n)]^2 \sim \sum_{i=1}^{n} n^{2\alpha_i}[\bar{F}_i(s)]^2 \leq \left(\sum_{i=1}^{n} n^{2\alpha_i}\right)[z(s)]^2
\end{aligned}$$

The variance of $Z^Y_{NR1}(s)$ can be verified similarly.

Proof of Proposition 4.10

Because $\Phi^{(n-2)}$ is differentiable, the survival distribution function of the radius R becomes

$$\bar{F}_R(x) = \sum_{j=0}^{n-1}(-1)^j \frac{x^j}{j!}\Phi^{(j)}(x)$$

Following the property of the regularly varying function, with $\Phi \in \mathcal{RV}_\infty(-\beta)$ then for $j = 1, \cdots, (n-1)$, we have

$$\lim_{x\to\infty}\frac{(-1)^j\ x^j\ \Phi^{(j)}(x)}{\Phi(x)} = \beta(\beta+1)\cdots(\beta+j-1)$$

and we can deduce

$$\lim_{x\to\infty}\frac{\bar{F}_R(x)}{\Phi(x)} = \lim_{x\to\infty}\frac{\sum_{j=1}^{n-1}(-1)^j\frac{x^j}{j!}\Phi^{(j)}(x)}{\Phi(x)} = \sum_{j=1}^{n-1}\frac{\beta(\beta+1)\cdots(\beta+j-1)}{j!}$$

We define $g(r) = \sum_{i=1}^{n}\bar{F}_i^{\leftarrow}(\Phi(r))$ and $L_0^Y(s) = \inf\{r \in \mathcal{R}^+ : g(r) \geq s\}$. Because $\bar{F}^{\leftarrow}$ and Φ are both non-increasing functions then for all $\mathbf{W} \in \mathbb{S}_n$ we have

$$g(r) = \sum_{i=1}^{n}\bar{F}_i^{\leftarrow}(\Phi(r\times 1)) \geq \sum_{i=1}^{n}\bar{F}_i^{\leftarrow}(\Phi(rW_i))$$

then $L_0^Y(s) \leq L^Y(\mathbf{W}, s)\ \forall \mathbf{W} \in \mathbb{S}_n$. Moreover, from the definiton of $L_0^Y(s)$, we have

$$\max_{i=1,2,\cdots,n} \bar{F}_i^{\leftarrow}(\Phi(L_0^Y(s))) \geq s/n \rightarrow \Phi(L_0^Y(s)) \leq \max_{i=1,2,\cdots,n} \bar{F}_i(s/n) \leq n^{\alpha_n} \max_{i=1,2,\cdots,n} \bar{F}_i(s) \leq n^{\alpha_n} z(s)$$

Thus, with $\lim_{s\to\infty} L_0^Y(s) = \infty$, the second moment of $Z_{NR2}(s)$ is bounded by

$$\begin{aligned}
\mathbb{E}(\left[Z_{NR2}^Y(s)\right]^2) &\leq 2\left(\left[P(M_n^Y > s)\right]^2 + \mathbb{E}\left(\left[\bar{F}_R(L^Y(\mathbf{W}, s))\right]^2\right)\right) \\
&\leq 2\left(\left[P(M_n^Y > s)\right]^2 + \left[\bar{F}_R(L_0^Y(s))\right]^2\right) \\
&\sim 2\left(\left[P(M_n^Y > s)\right]^2 + \left[\sum_{j=1}^{n-1} \frac{\beta(\beta+1)\cdots(\beta+j-1)}{j!}\right]^2 [\Phi(L_0^Y(s))]^2\right) \\
&\leq 2\left([z(s)]^2 + \left[\sum_{j=1}^{n-1} \frac{\beta(\beta+1)\cdots(\beta+j-1)}{j!}\right]^2 \times n^{2\alpha_n}[z(s)]^2\right) \\
&\leq 2\left(1 + n^{2\alpha_n}\left[\sum_{j=1}^{n-1} \frac{\beta(\beta+1)\cdots(\beta+j-1)}{j!}\right]^2\right)[z(s)]^2
\end{aligned}$$

Proof of Proposition 4.11

We start with an inequation between Φ and F_R.

Remark *If Φ is a n-monotone function, $(n-1)$-times differentiable and the random variable R has the distribution function satisfies 4.6 then we have*

$$\frac{\Phi(ax)}{(1-a)^{(n-1)}} \geq \bar{F}_R(x)\ \forall x \in \mathcal{R}^+ \text{ and } a \in (0,1)$$

Indeed, because Φ is non-increasing function then there exists $\mu \in (ax, x)$ such that

$$\Phi(ax) = \sum_{k=0}^{n-2}(1-a)^k \frac{x^k}{k!}(-1)^k\Phi^{(k)}(x) + (1-a)^{(n-1)} \frac{x^{(n-1)}}{(n-1)!}(-1)^{(n-1)}\ \Phi^{(n-1)}(\mu)$$

Following the property of n-monotone function: $(-1)^{(n-2)}\Phi^{(n-2)}(x)$ is a convex function, then $(-1)^{(n-1)}\Phi^{(n-1)}(x)$ is a non-increasing function, that means $(-1)^{(n-1)}\Phi^{(n-1)}(\mu) \geq (-1)^{(n-1)}\Phi^{(n-1)}(x)$

because $\mu \leq x$. Thus we have

$$\Phi(ax) \geq \sum_{k=0}^{n-1}(1-a)^k(-1)^k \frac{x^k}{k!}\Phi^{(k)}(x)$$
$$\frac{\Phi(ax)}{(1-a)^{(n-1)}} \geq \sum_{k=0}^{n-1}(1-a)^{(k-n+1)}(-1)^k\frac{x^k}{k!}\Phi^{(k)}(x) \geq \bar{F}_R(x)$$

To verify the Proposition 4.11, first $Z^Y_{NR3,2}(s)$ is bounded by $\bar{F}_R\big(L^Y_\lambda(\mathbf{W},s)\big)$

$$Z^Y_{NR3,2}(s) = F_R\big(U^Y(\mathbf{W},s)\big) - F_R\left(L^Y_\lambda(\mathbf{W},s)\right) \leq \bar{F}_R\big(L^Y_\lambda(\mathbf{W},s)\big)$$

Moreover, from the definition of $\bar{F}_R\big(L^Y_\lambda(\mathbf{W},s)\big)$ and $M_2\{\dfrac{\Phi^{\leftarrow}(\bar{F}_i(\lambda s))}{W_i}\}$, there exists two indexes $i_1, i_2 \in 1,2,\cdots,n$ such that

$$L^Y_\lambda(\mathbf{W},s) \geq M_2\{\frac{\Phi^{\leftarrow}(\bar{F}_i(\lambda s))}{W_i}\} = \frac{\Phi^{\leftarrow}(\bar{F}_{i_1}(\lambda s))}{W_{i_1}} \vee \frac{\Phi^{\leftarrow}(\bar{F}_{i_2}(\lambda s))}{W_{i_2}}$$

Therefore,

$$\begin{cases} W_{i_1}L^Y_\lambda(\mathbf{W},s) \geq \Phi^{\leftarrow}(\bar{F}_{i_1}(\lambda s)) \\ W_{i_2}L^Y_\lambda(\mathbf{W},s) \geq \Phi^{\leftarrow}(\bar{F}_{i_2}(\lambda s)) \end{cases}$$
$$\rightarrow \begin{cases} \Phi\big(W_{i_1}L^Y_\lambda(\mathbf{W},s)\big) \leq \bar{F}_{i_1}(\lambda s) \\ \Phi\big(W_{i_2}L^Y_\lambda(\mathbf{W},s)\big) \leq \bar{F}_{i_2}(\lambda s) \end{cases}$$

Appling the remark 3.3 for $a = W_{i_j}; j = 1,2$ and $x = L^Y_\lambda(\mathbf{W},s)$

$$\begin{cases} (1-W_{i_1})^{n-1}\bar{F}_R\big(L^Y_\lambda(\mathbf{W},s)\big) \leq \Phi\big(W_{i_1}L^Y_\lambda(\mathbf{W},s)\big) \\ (1-W_{i_2})^{n-1}\bar{F}_R\big(L^Y_\lambda(\mathbf{W},s)\big) \leq \Phi\big(W_{i_2}L^Y_\lambda(\mathbf{W},s)\big) \end{cases}$$

and for $j = 1,2$ we have $\bar{F}_{i_j}(\lambda s) \sim \lambda^{-\alpha_{i_j}}\bar{F}_{i_j}(s) \leq \lambda^{-\alpha_n}z(s)$. Finally,

$$\begin{aligned} \mathbb{E}([Z^Y_{NR3,2}(s)]^2) \leq \mathbb{E}([\bar{F}_R\big(L^Y_\lambda(\mathbf{W},s)\big)]^2) &\leq \mathbb{E}\left(\left[(1-W_{i_1})^{-(n-1)} \wedge (1-W_{i_2})^{-(n-1)}\right]^2\right)\lambda^{-2\alpha_n}[z(s)]^2 \\ &\leq 2^{2n-2}\lambda^{-2\alpha_n}[z(s)]^2 \end{aligned}$$

Proof of Proposition 4.12

From the multivariate distribution function of $\mathbf{Y} = (Y_1,\cdots,Y_n)$ in the first section

$$\mathbb{P}(Y_1 \leq y_1,\cdots,Y_n \leq y_n) = \sum_{1\leq i_1,\cdots,i_j\leq n}(-1)^j\ C(\bar{F}_{i_1}(y_{i_1}),\cdots,\bar{F}_{i_j}(y_{i_j}))$$

We can calculate the derivative of $F(y_1, \cdots, y_n)$ following $\mathbf{y}_{-i}$. Note that in the sum of 2^n elements, there are only two elements are different from 0 after taking the derivatives $(n-1)$ times.

$$\begin{aligned}\frac{\delta^{(n-1)}\mathbb{P}(Y_1 \leq y_1, \cdots, Y_n \leq y_n)}{\delta y_1 \cdots \delta y_{i-1}\delta y_{i+1} \cdots \delta y_n} &= \Phi^{(n-1)}(\sum_{j\neq i}\Phi^{\leftarrow}(\bar{F}_j(y_j)))\prod_{j\neq i}\left(\Phi^{\leftarrow}\right)^{(1)}(\bar{F}_j(y_j))\prod_{j\neq i}f_j(y_j)\\ &- \Phi^{(n-1)}(\sum_{j=1}^{n}\Phi^{\leftarrow}(\bar{F}_j(y_j)))\prod_{j\neq i}\left(\Phi^{\leftarrow}\right)^{(1)}(\bar{F}_j(y_j))\prod_{j\neq i}f_j(y_j)\end{aligned}$$

and note that the density of $\mathbf{Y}_{-i}$ is

$$f(\mathbf{y}_{-i}) = \Phi^{(n-1)}(\sum_{j=1,j\neq i}^{n}\Phi^{\leftarrow}(\bar{F}_j(y_j)))\prod_{j=1,j\neq i}^{n}\left(\Phi^{\leftarrow}\right)^{(1)}(\bar{F}_j(y_j))\prod_{j=1,j\neq i}^{n}f_j(y_j)$$

The conditional distribution of $Y_i^* = Y_i|\mathbf{Y}_{-i} = \mathbf{y}_{-i}$ is then

$$\begin{aligned}\mathbb{P}(Y_i < y_i|\mathbf{Y}_{-i} = \mathbf{y}_{-i}) &= 1 - \frac{(-1)^{n-1}\Phi^{(n-1)}(\sum_{j=1}^{n}\Phi^{\leftarrow}(\bar{F}_j(y_j)))\prod_{j\neq i}\left(\Phi^{\leftarrow}\right)^{(1)}(\bar{F}_j(y_j))\prod_{j\neq i}f_j(y_j)}{(-1)^{n-1}\Phi^{(n-1)}(\sum_{j\neq i}\Phi^{\leftarrow}(\bar{F}_j(y_j)))\prod_{j\neq i}\left(\Phi^{\leftarrow}\right)^{(1)}(\bar{F}_j(y_j))\prod_{j\neq i}f_j(y_j)}\\ &= 1 - \frac{\Phi^{(n-1)}\big(\sum_{j=1}^{n}\Phi^{\leftarrow}(\bar{F}_j(y_j))\big)}{\Phi^{(n-1)}\big(\sum_{j\neq i}\Phi^{\leftarrow}(\bar{F}_j(y_j))\big)}\end{aligned}$$

Proof of Proposition 4.13

$$\mathbb{P}(S_n^Y > s, M_n^Y \leq \kappa s) = \mathbb{P}(S_n^Y > s, M_n^Y \leq \kappa s, M_{n-1}^Y > \frac{1-\kappa}{n-1}\, s)$$

If we estimate this probability conditionally on $\mathbf{W} \in \mathbb{S}_n$ by the same method of estimating $Z_{NR3,2}^Y(s)$, the value of λ in this case is $\frac{1-\kappa}{n-1} \in (0, 1/n)$, the second moment of this estimator is upper bounded by $2^{2n-2}\left(\frac{1-\kappa}{n-1}\right)^{-2\alpha_n} \times [z^Y(s)]^2$. Thus, the variance of $Z_{NR3,2}^Y(s)$ is bounded

by

$$\begin{aligned}
\mathbb{V}ar(Z^Y_{NR4}(s)) &\leq 2\sum_{i=1}^{n}[\left(\bar{F}_i(\kappa s) - \bar{F}_i(s)\right)]^2\mathbb{V}ar\left(\mathbb{I}_{\{S_n^{Y\kappa i}>s, Y_i^{\kappa i}=M_n^{Y\kappa i}\}}\right) + 2^{2n-1}\left(\frac{1-\kappa}{n-1}\right)^{-2\alpha_n}[z^Y(s)]^2 \\
&\leq 2\sum_{i=1}^{n}[\bar{F}_i(\kappa s)]^2 + 2^{2n-1}\left(\frac{1-\kappa}{n-1}\right)^{-2\alpha_n}[z^Y(s)]^2 \\
&\leq \left[2\kappa^{-2\alpha_n} + 2^{2n-1}\left(\frac{1-\kappa}{n-1}\right)^{-2\alpha_n}\right][z^Y(s)]^2
\end{aligned}$$

TABLE 4.1: Quality of our estimators, B.r.e: Bounded relative error

Estimator	Z_{NR1}	Z_{NR2}	Z_{NR3}	Z_{NR4}
Arch copula	B.r.e	Not verified	Not verified	Not verified
Arch survival copula	B.r.e	B.r.e ($\Phi \in \mathcal{RV}_\infty(-\beta)$)	Not verified	B.r.e
Other structure	B.r.e	Not verified	Not verified	Not verified

TABLE 4.2: Sum of 5 Lomax (2.5), s = 20, Kendall's $\tau = 0.1$

Copulas	$\mathbb{E}(Z_{NR1})$	$\mathbb{E}(Z_{NR2})$	$\mathbb{E}(Z_{NR3})$	$\mathbb{E}(Z_{NR4})$
	$e(Z_{NR1})$	$e(Z_{NR2})$	$e(Z_{NR3})$	$e(Z_{NR4})$
Clayton	0.00379306	0.00386807	0.00384904	0.00383534
	2.805	*3.174*	*1.916*	*0.923*
Gumbel	0.00734014	0.00722742	0.00711167	0.00718644
	2.818	*1.220*	*0.946*	*0.666*
Survival Clayton	0.00751367	0.00765628	0.00771573	0.007658015
	2.774	*0.088*	*0.095*	*0.145*
Survival Gumbel	0.00443284	0.00431637	0.00432776	0.004326611
	2.916	*0.297*	*0.295*	*0.191*

TABLE 4.3: Sum of 5 Lomax (2.5), s = 20, Kendall's $\tau = 0.5$

Copulas	$\mathbb{E}(Z_{NR1})$	$\mathbb{E}(Z_{NR2})$	$\mathbb{E}(Z_{NR3})$	$\mathbb{E}(Z_{NR4})$
	$e(Z_{NR1})$	$e(Z_{NR2})$	$e(Z_{NR3})$	$e(Z_{NR4})$
Clayton	0.00592816	0.00626043	0.006108816	0.00610554
	2.868	*2.132*	*1.696*	*0.829*
Gumbel	0.01522982	0.01551193	0.015515734	0.015427433
	2.106	*0.229*	*0.229*	*0.184*
Survival Clayton	0.01639236	0.01661593	0.016648815	0.016583226
	2.034	*0.112*	*0.109*	*0.126*
Survival Gumbel	0.01095976	0.01116367	0.011170268	0.011168049
	2.378	*0.065*	*0.065*	*0.110*

TABLE 4.4: Sum of 5 Lomax (2.5), s = 20, Kendall's $\tau = 0.9$

Copulas	$\mathbb{E}(Z_{NR1})$	$\mathbb{E}(Z_{NR2})$	$\mathbb{E}(Z_{NR3})$	$\mathbb{E}(Z_{NR4})$
	$e(Z_{NR1})$	$e(Z_{NR2})$	$e(Z_{NR3})$	$e(Z_{NR4})$
Clayton	0.01701347	0.01722898	0.01730004	0.01714553
	1.909	*0.636*	*0.635*	*0.363*
Gumbel	0.01818365	0.01918613	0.0191801	0.01919531
	1.922	*0.028*	*0.029*	*0.032*
Survival Clayton	0.0175496	0.01786598	0.01786844	0.01787014
	1.966	*0.017*	*0.017*	*0.023*
Survival Gumbel	0.01682336	0.01763007	0.01764852	0.01765862
	1.995	*0.088*	*0.089*	*0.091*

TABLE 4.5: Sum of 5 Lomax (2.5), s = 200, Kendall's $\tau = 0.1$

Copulas	$\mathbb{E}(Z_{NR1})$	$\mathbb{E}(Z_{NR2})$	$\mathbb{E}(Z_{NR3})$	$\mathbb{E}(Z_{NR4})$
	$e(Z_{NR1})$	$e(Z_{NR2})$	$e(Z_{NR3})$	$e(Z_{NR4})$
Clayton	9.27623E-06	9.10775E-06	9.07197E-06	9.07119E-06
	1.701	*3.396*	*0.480*	*0.270*
Gumbel	3.19991E-05	3.16144E-05	3.16638E-05	3.13636E-05
	3.216	*0.752*	*0.626*	*0.584*
Survival Clayton	2.1843E-05	2.18073E-05	2.17775E-05	2.17643E-05
	3.582	*0.130*	*0.165*	*0.145*
Survival Gumbel	9.18493E-06	9.2342E-06	9.22524E-06	9.22872E-06
	1.568	*0.130*	*0.070*	*0.112*

TABLE 4.6: Sum of 5 Lomax (2.5), s = 200, Kendall's $\tau = 0.5$

Copulas	$\mathbb{E}(Z_{NR1})$	$\mathbb{E}(Z_{NR2})$	$\mathbb{E}(Z_{NR3})$	$\mathbb{E}(Z_{NR4})$
	$e(Z_{NR1})$	$e(Z_{NR2})$	$e(Z_{NR3})$	$e(Z_{NR4})$
Clayton	9.54965E-06	9.24395E-06	9.37001E-06	9.37181E-06
	2.023	*2.882*	*0.182*	*0.120*
Gumbel	7.84723E-05	7.9696E-05	7.91973E-05	7.96030E-05
	2.148	*0.273*	*0.240*	*0.230*
Survival Clayton	8.67011E-05	8.63854E-05	8.60928E-05	8.61954E-05
	2.036	*0.111*	*0.113*	*0.133*
Survival Gumbel	1.49943E-05	1.53712E-05	1.53317E-05	1.53699E-05
	3.559	*0.263*	*0.274*	*0.179*

TABLE 4.7: Sum of 5 Lomax (2.5), s = 200, Kendall's $\tau = 0.9$

Copulas	$\mathbb{E}(Z_{NR1})$	$\mathbb{E}(Z_{NR2})$	$\mathbb{E}(Z_{NR3})$	$\mathbb{E}(Z_{NR4})$
	$e(Z_{NR1})$	$e(Z_{NR2})$	$e(Z_{NR3})$	$e(Z_{NR4})$
Clayton	1.0871E-05	1.1563E-05	1.03202E-05	1.07798E-05
	2.867	*6.739*	*1.077*	*0.818*
Gumbel	9.18427E-05	1.09482E-04	1.07552E-04	1.09196E-04
	1.973	*0.134*	*0.096*	*0.127*
Survival Clayton	1.14134E-04	9.27657E-05	9.27742E-05	9.27769E-05
	1.721	*0.017*	*0.017*	*0.015*
Survival Gumbel	7.74801E-05	7.93332E-05	7.87621E-05	7.94657E-05
	2.155	*0.136*	*0.198*	*0.139*

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Conclusion

Dans ce travail de thèse, nous avons apporté des contributions supplémentaires aux techniques de calcul et de simulation pour l'approximation de la queue de distribution de sommes des variables aléatoires à variations régulières.

Le premier chapitre de la thèse est constitué d'une introduction générale avec une présentation des principaux résultats obtenus.

Dans le deuxième chapitre, nous avons proposé un algorithme de calcul pour déterminer la queue de distribution d'une somme de variables aléatoires de type Pareto. Les variables n'ont pas nécessairement le même indice de queue ou le même paramètre d'échelle. Le résultat sous forme de développement en série est simple et facile à implémenter.

Dans le troisième chapitre, nous avons abordé la problématique de la simulation de la probabilité que la somme de variables aléatoires indépendantes à variations régulières soit plus grande qu'un seuil élevé. Les résultats obtenus sont plus efficaces que les méthodes précédemment proposées dans la littérature.

Le quatrième chapitre de cette thèse a apporté des éléments de réponse au problème de calcul de la queue de distribution de sommes de variables aléatoires dépendantes à variations régulières. La dépendance est modélisée sous forme de copule archimédienne. Par des techniques de Monte Carlo conditionnelles, nous proposons quatre estimateurs dont les erreurs relatives sont efficaces dans le contexte de simulations des événements rares. Les simulations à la fin du chapitre et dans l'Annexe B montrent que certains estimateurs sont également précis dans les cas des variables qui ne sont pas à variations régulières.

Appendix A

Some useful concepts

A.1 Properties of regularly varying function

Theorem A.1 (Representation theorem)**.** *A positive measurable function L on $[x_0, \infty)$ is slowly varying if and only if it can be written in the form*

$$L(x) = c(x) \exp\left\{ \int_{x_0}^{x} \frac{\epsilon(y)}{y} dy \right\}$$

where $c(.)$ is a measurable non-negative function such that $\lim_{x\to\infty} = c_0 \in [0, \infty)$ and $\epsilon(x) \to 0$ when $x \to \infty$.

From the representation theorem it is clear that a regularly varying function f with index α has representation

$$f(x) = x^{\alpha} c(x) \exp\left\{ \int_{x_0}^{x} \frac{\epsilon(y)}{y} dy \right\}$$

where $c(.)$ and $\epsilon(.)$ are as above.

Theorem A.2 (Uniform convergent)**.** *If f is regularly varying with index α (in the case $\alpha > 0$, assuming f bounded on each interval $(0, x]$), then for $0 < a < b < \infty$*

$$\lim_{x\to\infty} \frac{f(xt)}{f(x)} = t^{\alpha}, \text{ uniformly in } t$$

- *on each $[a, b]$ if $\alpha = 0$*
- *on each $(0, b]$ if $\alpha > 0$*

- *on each* $[a, \infty)$ *if* $\alpha < 0$

Theorem A.3 (Karamata theorem)**.** *Let* L *be slowly varying and locally bounded in* $[x_0, \infty)$ *for some* $x_0 \geq 0$*. Then*

- *for* $\alpha > -1$

$$\int_{x_0}^{x} t^{\alpha} L(t)dt \sim (\alpha + 1)^{-1} x^{\alpha+1} L(x)$$

- *for* $\alpha < -1$

$$\int_{x}^{\infty} t^{\alpha} L(t)dt \sim -(\alpha + 1)^{-1} x^{\alpha+1} L(x)$$

The conclusions of Karamata's theorem can alternatively be formulated as follows. Supppose f is regularly varying with index α and f is locally bounded on $[x_0, \infty)$ for some $x > 0$. Then

- for $\alpha > -1$

$$\lim_{x\to\infty} \frac{\int_{x_0}^{x} f(t)dt}{xf(x)} = \frac{1}{1+\alpha}$$

- for $\alpha < -1$

$$\lim_{x\to\infty} \frac{\int_{x_0}^{x} f(t)dt}{xf(x)} = -\frac{1}{1+\alpha}$$

A.2 Modelling dependence with copulas

Copula is a multivariate distribution function deffined on the unit cube $[0, 1]^n$, with uniformly distributed marginals. Sklar's theorem (1959) is the most important result regarding copulas, and is used in essentially all applications of copulas.

Theorem A.4 (Sklar, 1959)**.** *Let* $\boldsymbol{F}$ *be an n-dimensional distribution function with margins* $F_1, \cdots, F_n$*. Then there exists an n-copula* C *such that for all* $\boldsymbol{x}$ *in* $\boldsymbol{R}^n$

$$\boldsymbol{F}(x_1, \cdots, x_n) = C(F_1(x_1), \cdots, F_n(x_n))$$

If $F_1, \cdots, F_n$ are all continuous, then C is unique; otherwise C is uniquel. Conversely, if C is an n-copula and $F_1, \cdots, F_n$ are distribution functions, then the function F deffined above is an n-dimensional distribution function with margins $F_1, \cdots, F_n$.

Copula is useful to define the new measures of dependence, concordance between random variables for example. Let (x, y) and $(\tilde{x}, \tilde{y})$ be two observations from a vector (X, Y) of continuous random variables. Then (x, y) and $(\tilde{x}, \tilde{y})$ are said to be concordant if $(x - \tilde{x})(y - \tilde{y}) > 0$, and discordant if $(x - \tilde{x})(y - \tilde{y}) > 0$.

Theorem A.5 (Nelsen, 1999)**.** *Let (X, Y) and $(\tilde{X}, \tilde{Y})$ be independent vectors of continuous random variables with joint distribution functions $\boldsymbol{F}$ and $\tilde{\boldsymbol{F}}$, respectively, with common margins F_1 (of X and $\tilde{X}$) and F_2 (of Y and $\tilde{Y}$). Let C and $\tilde{C}$ denote the copulas of (X, Y) and $(\tilde{X}, \tilde{Y})$. Let Q denote the difference between the probability of concordance and discordance of (X, Y) and $(\tilde{X}, \tilde{Y})$, i.e. let*

$$Q = \mathbb{P}\left((X - \tilde{X})(Y - \tilde{Y}) > 0\right) - \mathbb{P}\left((X - \tilde{X})(Y - \tilde{Y}) < 0\right)$$

then

$$Q = Q(C, \tilde{C}) = 4 \int_0^1 \int_0^1 \tilde{C}(u, v)\ dC(u, v) - 1$$

Kendall's Tau and Spearman's rho are measures of concordance

$$\begin{aligned} \tau(X, Y) &= Q(C, C) = 4 \int_0^1 \int_0^1 C(u, v)\ dC(u, v) - 1 \\ \rho(X, Y) &= 3 \times Q(C, \Pi) = 12 \int_0^1 \int_0^1 C(u, v)\ dudv - 3 \end{aligned}$$

where Π is the independent copula $\Pi(u, v) = uv$.

Tail dependence is a concept that is relevant for the study of dependence between extreme values.

Definition A.6. *Let (X, Y) be a vector of continuous random variables with marginal distribution functions F_1 and F_2. The coefficient of upper tail dependence and lower tail dependence of (X, Y) are*

$$\begin{aligned} \lambda_U &= \lim_{u \to 1} \mathbb{P}(Y > F_2^{\leftarrow}(u) | X > F_1^{\leftarrow}(u)) \\ \lambda_L &= \lim_{u \to 0} \mathbb{P}(Y \leq F_2^{\leftarrow}(u) | X \leq F_1^{\leftarrow}(u)) \end{aligned}$$

provided that the limits λ_U and $\lambda_L \in [0, 1]$ exist.

Note that if the copula of (X, Y) is C and λ_U, λ_L exist then

$$\begin{aligned}\lambda_U &= \lim_{u\to 1} \frac{1-2u+C(u,u)}{1-u} \\ \lambda_L &= \lim_{u\to 0} \frac{C(u,u)}{u}\end{aligned}$$

A.3 Inverse Laplace transform and GWR alorithm

The analytic solution to the inverse problem is provided by the Post-Widder formula.

Theorem A.7. *Post-Widder formula Let $f(t)$ be a continuous function on the interval $[0, \infty)$ of exponential order, i.e. $\sup_{t>0} \frac{f(t)}{e^{ct}} < \infty$ for some real number c. Then for all $t \geq c$ the Laplace transform $\mathcal{L}(f(t))$ is existed and is infinitely differentiable. Furthermore, the inverse Laplace transform of $\mathcal{L}(f(t))$ is given by*

$$f(t) = \lim_{k\to\infty} \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} \mathcal{L}^{(k)}(f(t))$$

for $t > 0$, where $\mathcal{L}^{(k)}(f(t))$ is k^{th} derivative of $\mathcal{L}(f(t))$.

To simplify the formula, let $g(t) = \mathcal{L}(f(t))$ and $f_k(t) = \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} g(t)$. Gaver (1960) used the difference operator to approximate $f_k(t)$ by g

$$f_k(t) = \frac{ak}{t}\, C_{2k}^k \sum_{j=0}^{k} (-1)^j\, C_k^j g(a(k+j)/t)$$

with $a = \log(2)$. The Gaver function $f_k(t)$ can be computed by a recursive algorithm

$$\begin{aligned}G_0^{(n)} &= \frac{an}{t} g(an/t),\ 1 \leq n \leq 2M \\ G_k^{(n)} &= (1+\frac{n}{k}) G_{k-1}^{(n)} - \frac{n}{k} G_{k-1}^{(n+1)},\ k \geq 1,\ n \geq k \\ f_k(t) &= G_k^{(k)}\end{aligned}$$

The Gaver functionals provide a poor approximation because $|f(t) - f_k(t)| \sim c/k$ as $k \to \infty$. The Gave-Wynn-Rho (GWR) algorithm is based on a special sequence acceleration of the Gaver functionals and is calculated by algorithm of Wynn rho:

- First set the precision to M where M is an even integer.
- Compute the function $f_1(t), f_2(t), \cdots, f_M(t)$ by A.1

- Approximation for $f(t)$: $f(t, M)$ is calculated recursively

$$\rho_{-1}^{(n)} = 0 \; ; \; \rho_{0}^{(n)} = f_n(t) \; ; \; \rho_{k}^{(n)} = \rho_{k-2}^{(n+1)} + \frac{k}{\rho_{k-1}^{(n+1)} - \rho_{k-1}^{(n)}}$$

- Return $f(t, M) = \rho_{M}^{(0)}$

Appendix B

Some numerical studies

TABLE B.1: Compound sum Pareto: Geometric($p = 1/3$); Binominal($p = 0.1, n = 30$), Poisson($\lambda = 3$)

s	$P(S_G > s)$	$P(S_B > s)$	$P(S_P > s)$
50	0.01490215	0.01116803	0.01131394
100	0.00369731	0.00343022	0.00344884
500	0.00027840	0.00027546	0.00027572
1000	0.00009661	0.00009611	0.00009616
VaR	S_G	S_B	S_P
99.5%	84.43124592	79.64140474	79.97917234
99.9%	220.3769147	216.64065718	216.9542133
$TVaR$	S_G	S_B	S_P
99.5%	100.4483012	87.6828265	88.3558543
99.9%	266.9683078	237.6730277	239.0591047

TABLE B.2: High precision, Compound Pareto ($\alpha = 1.38673, \beta = 1$)

$s = 75$	Compound Poisson, $P(S_P \leq s)$, $\lambda_P = 3$
K=150	0,99090,62583,94987,94461,16572,31656,15350,14832,32813,94395,01447,26730
K=100	0,99090,62583,94987,94461,16572,31656,15350,14832,32813,94395,01447,26730
K=50	0,99090,62583,94987,94461,16572,31656,15351,00086,49012,35983,14979,80984
K=30	0,99090,62583,94987,94461,16572,31628,64023,20306,96016,83345,08914,14328
K=20	0,99090,62583,94987,94462,56454,79154,48504,15251,02565,94152,33592,34410
K=10	0,99090,62583,90735,45308,51126,61397,07815,12376,91297,91588,92462,66065
$s = 100$	Compound Poisson, $P(S_P \leq s)$, $\lambda_P = 3$
K=150	0.99417,65191,32201,23756,51304,00023,56082,69776,56939,81749,87515,26844
K=100	0.99417,65191,32201,23756,51304,00023,56082,69776,56939,81749,87515,26844
K=50	0.99417,65191,32201,23756,51304,00023,56082,69776,56954,57572,41854,76785
K=30	0.99417,65191,32201,23756,51304,00023,54296,56972,21153,10436,14462,46722
K=20	0.99417,65191,32201,23756,51825,31202,03063,85905.94601,91372,48430,61888
K=10	0.99417,65191,32013,84094,16956,07007,45727,97039.87378,67188,89473,56658
$s = 75$	Compound Geometric, $P(S_G \leq s)$, $p_G = 0.25$
K=150	0.98783,48893,23327,92839,77666,26575,71893,51597,49148,29564,64519,99008
K=100	0.98783,48893,23327,92839,77666,26508,91804,68002,48064,62555,06067,59478
s = 100	Compound Geometric, $P(S_G \leq s)$, $p_G = 0.25$
K=150	0.99285,99202,94532,95196,44116,37967,78586,42958,78302,42448,63054,96633
K=100	0.99285,99202,94532,95196,44116,37967,78586,42958,78457,31388,16320,28661

TABLE B.3: High precision, VaR, TVaR of Compound Poisson($\lambda_P = 3$) Pareto($\alpha = 1.38673, \beta = 1$)

$VaR_{0.99}(S_P)$	70.62034,28949,69582,71093,65490
$Error \leq$	10^{-28}
$TVaR_{0.99}(S_P)$	229.10141,44343,25739,00403,21500
$Error \leq$	10^{-28}

TABLE B.4: High precision, VaR, TVaR of Compound Geometric($p_G = 1/3$) Pareto($\alpha = 1.38673, \beta = 1$)

$VaR_{0.99}(S_G)$	83.18821,26447,33167,08707,11829
$Error \leq$	10^{-32}
$TVaR_{0.99}(S_G)$	240.98320,30652,37699,15345,55790
$Error \leq$	10^{-32}

TABLE B.5: Sum of 50 Exps($\mu = 1$), survival Clayton, Kendall's $\tau = 1/3$, simulated by Z_{NR2}

s	$P(S_{50} \leq s)$	$e(Z_{NR2})$
100	0.097574172	0.0721066
200	0.010941223	0.0973476
300	0.001425116	0.1070046
400	0.000191641	0.1095835
500	2.59119e-05	0.1104851

TABLE B.6: Sum of 50 Paretos($\alpha = 2.5, \beta = 1$), survival Clayton, Kendall's $\tau = 0.4$, simulated by Z_{NR2}

s	$P(S_{50} \leq s)$	$e(Z_{NR2})$
1000	0.009473277	0.03167215
5000	0.0008963652	0.03179081
20,000	1.132780e-04	0.03212107
100,000	1.015752e-05	0.03195867

TABLE B.7: Sum of 50 Paretos($\alpha = 2.5, \beta = 1$), survival Clayton, Kendall's $\tau = 0.7$, simulated by Z_{NR2}

s	$P(S_{50} \leq s)$	$e(Z_{NR2})$
1000	0.0102844634	0.01844034
5000	0.0009747038	0.01836228
20,000	1.232478e-04	0.01857353
100,000	1.105643e-05	0.01843685